

MODULES OVER QUANTIZED COORDINATE ALGEBRAS AND PBW-BASES

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ABSTRACT. Around 1990 Soibelman constructed certain irreducible modules over the quantized coordinate algebra. A. Kuniba, M. Okado, Y. Yamada [8] recently found that the relation among natural bases of Soibelman's irreducible module can be described using the relation among the PBW-type bases of the positive part of the quantized enveloping algebra, and proved this fact using case-by-case analysis in rank two cases. In this paper we will give a realization of Soibelman's module as an induced module, and give a unified proof of the above result of [8]. We also verify Conjecture 1 of [8] about certain operators on Soibelman's module.

1. INTRODUCTION

1.1. Let G be a connected simply-connected simple algebraic group over the complex number field \mathbb{C} with Lie algebra \mathfrak{g} . The coordinate algebra $\mathbb{C}[G]$ of G is a Hopf algebra which is dual to the enveloping algebra $U(\mathfrak{g})$ of \mathfrak{g} . So we can naturally define a q -analogue $\mathbb{C}_q[G]$ of $\mathbb{C}[G]$ as the Hopf algebra dual to the quantized enveloping algebra $U_q(\mathfrak{g})$. This paper is concerned with the representation theory of the quantized coordinate algebra $\mathbb{C}_q[G]$.

Since the ordinary coordinate algebra $\mathbb{C}[G]$ is commutative, its irreducible modules are all one-dimensional and are in one-to-one correspondence with the points of G ; however, the quantized coordinate algebra $\mathbb{C}_q[G]$ is non-commutative, and its representation theory is much more complicated. In fact, Soibelman [12] already pointed out around 1990 that there are not so many one-dimensional $\mathbb{C}_q[G]$ -modules and that there really exist infinite dimensional irreducible $\mathbb{C}_q[G]$ -modules.

Let us recall Soibelman's result more precisely. He considered the situation where the parameter q is a positive real number with $q \neq 1$. In this case $\mathbb{C}_q[G]$ is endowed with a structure of $*$ -algebra, and we have the notion of unitarizable $\mathbb{C}_q[G]$ -modules. Soibelman showed that one-dimensional unitarizable $\mathbb{C}_q[G]$ -modules are in one-to-one correspondence with the points of the maximal compact subgroup H_{cpt} of the maximal torus H of G . Denote the one-dimensional $\mathbb{C}_q[G]$ -module corresponding to $h \in H_{\text{cpt}}$ by \mathbb{C}_h . On the other hand infinite-dimensional irreducible unitarizable $\mathbb{C}_q[G]$ -modules are constructed as follows. In the case $G = SL_2$ Vaksman and Soibelman [14] constructed an irreducible unitarizable $\mathbb{C}_q[SL_2]$ -modules \mathcal{F} with basis $\{m_n\}_{n \in \mathbb{Z}, n \geq 0}$ using an explicit description of $\mathbb{C}_q[SL_2]$. For general G denote by I the index set of simple roots. For each $i \in I$ we have a natural Hopf

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algebra homomorphism $\pi_i : \mathbb{C}_q[G] \rightarrow \mathbb{C}_{q_i}[SL_2]$, where q_i is some power of q . Via π_i we can regard \mathcal{F} as a $\mathbb{C}_q[G]$ -module. Denote this $\mathbb{C}_q[G]$ -module by \mathcal{F}_i . Let W be the Weyl group of G . For $w \in W$ we denote the length of w by $\ell(w)$. Take $w \in W$ and its reduced expression $w = s_{i_1} \cdots s_{i_{\ell(w)}} \ (i_r \in I)$ as a product of simple reflections. Soibelman proved that the tensor product $\mathcal{F}_{i_1} \otimes \cdots \otimes \mathcal{F}_{i_{\ell(w)}}$ is a unitarizable irreducible $\mathbb{C}_q[G]$ -module. Moreover, he showed that $\mathcal{F}_{i_1} \otimes \cdots \otimes \mathcal{F}_{i_{\ell(w)}}$ depends only on w . So we can denote this $\mathbb{C}_q[G]$ -module by \mathcal{F}_w . It is also verified in [12] that any irreducible unitarizable $\mathbb{C}_q[G]$ -module is isomorphic to the tensor product $\mathcal{F}_w \otimes \mathbb{C}_h$ for $w \in W$, $h \in H_{\text{cpt}}$.

As for further development of the theory of $\mathbb{C}_q[G]$ -modules we refer to Joseph [4], Yakimov [15].

Quite recently the above work of Soibelman has been taken up again by Kuniba, Okado, Yamada [8]. Let $w_0 \in W$ be the longest element. Note that for each reduced expression $w_0 = s_{i_1} \cdots s_{i_{\ell(w_0)}}$ of w_0 we have a basis

$$\mathcal{B}_{i_1, \dots, i_{\ell(w_0)}} = \{m_{n_1} \otimes \cdots \otimes m_{n_{\ell(w_0)}} \mid n_1, \dots, n_{\ell(w_0)} \geq 0\}$$

of $\mathcal{F}_{w_0} = \mathcal{F}_{i_1} \otimes \cdots \otimes \mathcal{F}_{i_{\ell(w_0)}}$ parametrized by the set of $\ell(w_0)$ -tuples $(n_1, \dots, n_{\ell(w_0)})$ of non-negative integers. On the other hand, by Lusztig's result, for each reduced expression $w_0 = s_{i_1} \cdots s_{i_{\ell(w_0)}}$ of w_0 we have a PBW-type basis $\mathcal{B}'_{i_1, \dots, i_{\ell(w_0)}}$ of the positive part $U_q(\mathfrak{n}^+)$ of $U_q(\mathfrak{g})$ parametrized by the set of $\ell(w_0)$ -tuples of non-negative integers. Kuniba, Okado, Yamada observed in [8] that for two reduced expressions $w_0 = s_{i_1} \cdots s_{i_{\ell(w_0)}} = s_{j_1} \cdots s_{j_{\ell(w_0)}}$ of w_0 the transition matrix between $\mathcal{B}_{i_1, \dots, i_{\ell(w_0)}}$ and $\mathcal{B}_{j_1, \dots, j_{\ell(w_0)}}$ coincides with the transition matrix between $\mathcal{B}'_{i_1, \dots, i_{\ell(w_0)}}$ and $\mathcal{B}'_{j_1, \dots, j_{\ell(w_0)}}$ up to a normalization factor. They proved this fact partly using a case-by-case argument in rank two cases.

In the present paper we give a new approach to the results of Soibelman [12] and Kuniba, Okado, Yamada [8]. We work over the rational function field $\mathbb{F} = \mathbb{Q}(q)$; however, our arguments also hold in a more general situation (see Section 8 below). Let $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ be the triangular decomposition of \mathfrak{g} . Let N^\pm and B^\pm be the subgroups of G corresponding to \mathfrak{n}^\pm and $\mathfrak{h} \oplus \mathfrak{n}^\pm$ respectively. For each $w \in W$ we define a $\mathbb{C}_q[G]$ -module $\overline{\mathcal{M}}_w$ as the induced module from a one-dimensional representation of a certain subalgebra $\mathbb{C}_q[(N^- \cap wN^+w^{-1}) \setminus G]$ of $\mathbb{C}_q[G]$. We will show that $\overline{\mathcal{M}}_w$ is an irreducible $\mathbb{C}_q[G]$ -module and that for each reduced expression $w = s_{i_1} \cdots s_{i_{\ell(w)}}$ we have a decomposition $\overline{\mathcal{M}}_w \cong \mathcal{F}_{i_1} \otimes \cdots \otimes \mathcal{F}_{i_{\ell(w)}}$ into tensor product. This gives a new proof of Soibelman's result. We will also show that there exists a natural linear isomorphism

$$(1.1) \quad \overline{\mathcal{M}}_w \cong U_q(\mathfrak{n}^+ \cap wn^-),$$

where $U_q(\mathfrak{n}^+ \cap wn^-)$ is a certain subalgebra of $U_q(\mathfrak{g})$ defined in terms of Lusztig's braid group action (see De Concini, Kac, Procesi [2], Lusztig [10]). From this we obtain (in the case $w = w_0$) the result of Kuniba, Okado, Yamada described above. As in [8] a certain localization of $\mathbb{C}_q[G]$ plays a crucial role in the proof. More precisely, for each $w \in W$ we consider the localization $\mathbb{C}_q[wN^+B^-]$ of $\mathbb{C}_q[G]$, which is a q -analogue of $\mathbb{C}[wN^+B^-]$. In addition to it, we use the Drinfeld pairing between the positive and negative

parts of the quantized enveloping algebra in constructing the isomorphism (1.1). A crucial difference between Soibelman's approach and our approach is that, instead of the decomposition

$$\mathbb{C}_q[G] = \mathbb{C}_q[G/N^+] \mathbb{C}_q[G/N^-]$$

used by Soibelman, we utilize the q -analogue of the decomposition

$$\mathbb{C}[B^- w_0 B^-] \cong \mathbb{C}[B^- w_0 B^- / B^-] \otimes \mathbb{C}[N^- \setminus B^- w_0 B^-]$$

in the case $w = w_0$, and

$$\begin{aligned} \mathbb{C}[wN^+ B^-] &\cong \mathbb{C}[(wN^+ w^{-1} \cap N^-)] \otimes \mathbb{C}[(wN^+ w^{-1} \cap N^+) wB^-] \\ &\cong \mathbb{C}[(wN^+ w^{-1} \cap N^-)] \otimes \mathbb{C}[(wN^+ w^{-1} \cap N^-) \setminus wN^+ B^-], \end{aligned}$$

for general w , which is more natural from geometric point of view. As a consequence of our approach, we can also show easily a conjecture of Kuniba, Okado, Yamada [8, Conjecture 1] concerning the action of a certain element of $\mathbb{C}_q[wN^+ B^-]$ on $\overline{\mathcal{M}}_w$.

We finally note that our results hold true for any symmetrizable Kac-Moody algebra (see Section 8 below). We hope this fact will be useful in the investigation of 3-dimensional integrable systems, which was the original motivation of [8]. After writing up the first draft of this paper Yoshiyuki Kimura pointed out to me that Proposition 2.10 below in the Kac-Moody case is not an obvious fact which is stated as a conjecture in Berenstein and Greenstein [1, Conjecture 5.5]. In the present manuscript we have included a proof of Proposition 2.10 which works for the Kac-Moody case. We heard that Kimura also proved it by a different method (see Kimura [6]).

After finishing this work we heard that Yoshihisa Saito [11] has obtained similar results by a different method.

1.2. We use the following notation for Hopf algebras throughout the paper. For a Hopf algebra H over a field \mathbb{K} we denote its multiplication, comultiplication, counit, antipode by $m_H : H \otimes_{\mathbb{K}} H \rightarrow H$, $\Delta_H : H \rightarrow H \otimes_{\mathbb{K}} H$, $\varepsilon_H : H \rightarrow \mathbb{K}$, $S_H : H \rightarrow H$ respectively. The subscript H is often omitted. For left H -modules V_0, \dots, V_m we regard $V_0 \otimes_{\mathbb{K}} \dots \otimes_{\mathbb{K}} V_m$ as a left H -module via the iterated comultiplication $\Delta_m : H \rightarrow H^{\otimes m+1}$. We will occasionally use Sweedler's notation for the comultiplication

$$\Delta(h) = \sum_{(h)} h_{(0)} \otimes h_{(1)} \quad (h \in H),$$

and the iterated comultiplication

$$\Delta_m(h) = \sum_{(h)_m} h_{(0)} \otimes \dots \otimes h_{(m)} \quad (h \in H).$$

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2. QUANTIZED ENVELOPING ALGEBRAS

2.1. Let G be a connected simply-connected simple algebraic group over the complex number field \mathbb{C} . We take Borel subgroups B^+ and B^- such that $H = B^+ \cap B^-$ is a maximal torus of G , and set $N^\pm = [B^\pm, B^\pm]$. The Lie algebras of G , B^\pm , H , N^\pm are denoted by \mathfrak{g} , \mathfrak{b}^\pm , \mathfrak{h} , \mathfrak{n}^\pm respectively. We denote by P the character group of H . Let Δ^+ and Δ^- be the subsets of P consisting of weights of \mathfrak{n}^+ and \mathfrak{n}^- respectively, and set $\Delta = \Delta^+ \cup \Delta^-$. Then Δ is the set of roots of \mathfrak{g} with respect to \mathfrak{h} . We denote by $\Pi = \{\alpha_i \mid i \in I\}$ the set of simple roots of Δ such that Δ^+ is the set of positive roots. Let P^+ be the set of dominant weights in P with respect to Π , and set $P^- = -P^+$. We set

$$Q = \sum_{i \in I} \mathbb{Z}\alpha_i, \quad Q^+ = \sum_{i \in I} \mathbb{Z}_{\geq 0}\alpha_i,$$

where $\mathbb{Z}_{\geq 0}$ denotes the set of non-negative integers. The Weyl group $W = N_G(H)/H$ naturally acts on P and Q . By differentiation we will regard P as a \mathbb{Z} -lattice of \mathfrak{h}^* in the following. We denote by

$$(\cdot, \cdot) : \mathfrak{h}^* \times \mathfrak{h}^* \rightarrow \mathbb{C}$$

the W -invariant non-degenerate symmetric bilinear form such that $(\alpha, \alpha) = 2$ for short roots α . For $\alpha \in \Delta$ we set $\alpha^\vee = 2\alpha/(\alpha, \alpha)$. As a subgroup of $GL(\mathfrak{h}^*)$ the Weyl group W is generated by the simple reflections s_i ($i \in I$) given by $s_i(\lambda) = \lambda - (\lambda, \alpha_i^\vee)\alpha_i$ ($\lambda \in \mathfrak{h}^*$). We denote by $\ell : W \rightarrow \mathbb{Z}_{\geq 0}$ the length function with respect to the generating set $\{s_i \mid i \in I\}$ of W . The longest element of W is denoted by w_0 . For $w \in W$ we set

$$\mathcal{I}_w = \{(i_1, \dots, i_{\ell(w)}) \in I^{\ell(w)} \mid w = s_{i_1} \cdots s_{i_{\ell(w)}}\}.$$

2.2. For $n \in \mathbb{Z}$ we set

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \in \mathbb{Z}[q, q^{-1}].$$

For $m \in \mathbb{Z}_{\geq 0}$ we set

$$[m]_q! = [m]_q [m-1]_q \cdots [1]_q.$$

For $m, n \in \mathbb{Z}$ with $m \geq 0$ we set

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{[n]_q [n-1]_q \cdots [n-m+1]_q}{[m]_q [m-1]_q \cdots [1]_q} \in \mathbb{Z}[q, q^{-1}].$$

For $i \in I$ we set $q_i = q^{(\alpha_i, \alpha_i)/2}$, and for $i, j \in I$ we further set $a_{ij} = (\alpha_i^\vee, \alpha_j)$.

We denote by $U = U_q(\mathfrak{g})$ the quantized enveloping algebra of \mathfrak{g} . Namely, it is an associative algebra over $\mathbb{F} = \mathbb{Q}(q)$ generated by the elements $k_i^{\pm 1}$, e_i ,

f_i ($i \in I$) satisfying the defining relations

$$\begin{aligned}
k_i k_i^{-1} &= k_i^{-1} k_i = 1 & (i \in I), \\
k_i k_j &= k_j k_i & (i, j \in I), \\
k_i e_j k_i^{-1} &= q_i^{a_{ij}} e_j & (i, j \in I), \\
k_i f_j k_i^{-1} &= q_i^{-a_{ij}} f_j & (i, j \in I), \\
e_i f_j - f_j e_i &= \delta_{ij} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}} & (i, j \in I), \\
\sum_{m=0}^{1-a_{ij}} (-1)^m e_i^{(1-a_{ij}-m)} e_j e_i^{(m)} &= 0 & (i, j \in I, i \neq j), \\
\sum_{m=0}^{1-a_{ij}} (-1)^m f_i^{(1-a_{ij}-m)} f_j f_i^{(m)} &= 0 & (i, j \in I, i \neq j),
\end{aligned}$$

where

$$e_i^{(m)} = \frac{1}{[m]_{q_i}!} e_i^m, \quad f_i^{(m)} = \frac{1}{[m]_{q_i}!} f_i^m \quad (m \in \mathbb{Z}_{\geq 0}).$$

We endow U with the Hopf algebra structure given by

$$\Delta(k_i^{\pm 1}) = k_i^{\pm 1} \otimes k_i^{\pm 1}, \quad \Delta(e_i) = e_i \otimes 1 + k_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes k_i^{-1} + 1 \otimes f_i,$$

$$\varepsilon(k_i^{\pm 1}) = 1, \quad \varepsilon(e_i) = \varepsilon(f_i) = 0,$$

$$S(k_i^{\pm 1}) = k_i^{\mp 1}, \quad S(e_i) = -k_i^{-1} e_i, \quad S(f_i) = -f_i k_i.$$

We define subalgebras $U^0 = U_q(\mathfrak{h})$, $U^+ = U_q(\mathfrak{n}^+)$, $U^- = U_q(\mathfrak{n}^-)$, $U^{\geq 0} = U_q(\mathfrak{b}^+)$, $U^{\leq 0} = U_q(\mathfrak{b}^-)$ by

$$U^0 = \langle k_i^{\pm 1} \mid i \in I \rangle, \quad U^+ = \langle e_i \mid i \in I \rangle, \quad U^- = \langle f_i \mid i \in I \rangle,$$

$$U^{\geq 0} = \langle k_i^{\pm 1}, e_i \mid i \in I \rangle, \quad U^{\leq 0} = \langle k_i^{\pm 1}, f_i \mid i \in I \rangle,$$

respectively. Then U^0 , $U^{\geq 0}$, $U^{\leq 0}$ are Hopf subalgebras. The multiplication of U induces isomorphisms

$$U \cong U^+ \otimes U^0 \otimes U^- \cong U^- \otimes U^0 \otimes U^+,$$

$$U^{\geq 0} \cong U^0 \otimes U^+ \cong U^+ \otimes U^0, \quad U^{\leq 0} \cong U^0 \otimes U^- \cong U^- \otimes U^0.$$

REMARK 2.1. In this paper $\otimes_{\mathbb{F}}$ is often written as \otimes .

For $\gamma = \sum_{i \in I} m_i \alpha_i \in Q$ we set

$$k_\gamma = \prod_{i \in I} k_i^{m_i} \in U^0.$$

Then we have $U^0 = \bigoplus_{\gamma \in Q} \mathbb{F} k_\gamma$, and hence U^0 is isomorphic to the group algebra of Q . For $\gamma \in Q^+$ we define $U_{\pm\gamma}^\pm$ by

$$U_{\pm\gamma}^\pm = \{u \in U^\pm \mid k_i u k_i^{-1} = q_i^{\pm(\alpha_i^\vee, \gamma)} u \ (i \in I)\}.$$

Then we have $U^\pm = \bigoplus_{\gamma \in Q^+} U_{\pm\gamma}^\pm$.

2.3. There exists a unique bilinear map

$$(2.1) \quad \tau : U^{\geq 0} \times U^{\leq 0} \rightarrow \mathbb{F}$$

characterized by the properties:

$$(2.2) \quad (\tau \otimes \tau)(\Delta(x), y_2 \otimes y_1) = \tau(x, y_1 y_2) \quad (x \in U^{\geq 0}, y_1, y_2 \in U^{\leq 0}),$$

$$(2.3) \quad (\tau \otimes \tau)(x_1 \otimes x_2, \Delta(y)) = \tau(x_1 x_2, y) \quad (x_1, x_2 \in U^{\geq 0}, y \in U^{\leq 0}),$$

$$(2.4) \quad \tau(e_i, k_\lambda) = \tau(k_\lambda, f_i) = 0 \quad (i \in I, \lambda \in Q),$$

$$(2.5) \quad \tau(k_\lambda, k_\mu) = q^{(\lambda, \mu)} \quad (\lambda, \mu \in Q),$$

$$(2.6) \quad \tau(e_i, f_j) = \delta_{ij} \frac{1}{q_i - q_i^{-1}} \quad (i, j \in I).$$

We call it the Drinfeld pairing. It also satisfies the following properties:

$$(2.7) \quad \tau(Sx, Sy) = \tau(x, y) \quad (x \in U^{\geq 0}, y \in U^{\leq 0}),$$

$$(2.8) \quad \tau(k_\lambda x, k_\mu y) = \tau(x, y) q^{(\lambda, \mu)} \quad (x \in U^+, y \in U^-),$$

$$(2.9) \quad \gamma, \delta \in Q^+, \gamma \neq \delta \implies \tau|_{U_\gamma^+ \times U_{-\delta}^-} = 0,$$

$$(2.10) \quad \gamma \in Q^+ \implies \tau|_{U_\gamma^+ \times U_{-\gamma}^-} \text{ is non-degenerate.}$$

2.4. For a U^0 -module M and $\lambda \in P$ we set

$$M_\lambda = \{m \in M \mid k_i m = q_i^{(\lambda, \alpha_i^\vee)} m \ (i \in I)\}.$$

We say that a U^0 -module M is a weight module if $M = \bigoplus_{\lambda \in P} M_\lambda$.

For a U -module V we regard $V^* = \text{Hom}_{\mathbb{F}}(V, \mathbb{F})$ as a right U -module by

$$\langle v^* u, v \rangle = \langle v^*, uv \rangle \quad (v \in V, v^* \in V^*, u \in U).$$

Denote by $\text{Mod}_0(U)$ (resp. $\text{Mod}_0^r(U)$) the category of finite-dimensional left (resp. right) U -modules which is a weight module as a U^0 -module. Here, a right U^0 -module M is regarded as a left U^0 -module by

$$tm := mt \quad (m \in M, t \in U^0).$$

If $V \in \text{Mod}_0(U)$, then we have $V^* \in \text{Mod}_0^r(U)$. This gives an anti-equivalence $\text{Mod}_0(U) \ni V \mapsto V^* \in \text{Mod}_0^r(U)$ of categories.

For $\lambda \in P^-$ we denote by $V(\lambda)$ the finite-dimensional irreducible (left) U -module with lowest weight λ . Namely, $V(\lambda)$ is a finite-dimensional U -module generated by a non-zero element $v_\lambda \in V(\lambda)_\lambda$ satisfying $f_i v_\lambda = 0$ ($i \in I$). Then $\text{Mod}_0(U)$ is a semisimple category with simple objects $V(\lambda)$ for $\lambda \in P^-$ (see Lusztig [10]). For $\lambda \in P^-$ we set $V^*(\lambda) = (V(\lambda))^*$, and define $v_\lambda^* \in V^*(\lambda)_\lambda$ by $\langle v_\lambda^*, v_\lambda \rangle = 1$.

The following well known fact will be used occasionally in this paper (see e.g. [13, Lemma 2.1]).

PROPOSITION 2.2. *Let $\gamma \in Q^+$.*

- (i) *For sufficiently small $\lambda \in P^-$ the linear map $U_\gamma^+ \ni x \mapsto x v_\lambda \in V(\lambda)_{\lambda+\gamma}$ is bijective.*
- (ii) *For sufficiently small $\lambda \in P^-$ the linear map $U_{-\gamma}^- \ni y \mapsto v_\lambda^* y \in V^*(\lambda)_{\lambda+\gamma}$ is bijective.*

REMARK 2.3. In this paper the expression “for sufficiently small $\lambda \in P^-$...” means that “there exists some $\mu \in P^-$ such that for any $\lambda \in \mu + P^-$...”.

2.5. For $i \in I$ and $M \in \text{Mod}_0(U)$ we denote by $\dot{T}_i, \hat{T}_i \in GL(M)$ the operators denoted by $T''_{i,1}$ and $T''_{i,-1}$ respectively in [10]. We have also algebra automorphisms \dot{T}_i, \hat{T}_i of U satisfying

$$\dot{T}_i(um) = \dot{T}_i(u)\dot{T}_i(m), \quad \hat{T}_i(um) = \hat{T}_i(u)\hat{T}_i(m)$$

for $u \in U, m \in M \in \text{Mod}_0(U)$. They are given by

$$\begin{aligned} \dot{T}_i(e_j) &= \begin{cases} -f_i k_i & (j = i) \\ \sum_{r=0}^{-a_{ij}} (-1)^r q_i^{-r} e_i^{(-a_{ij}-r)} e_j e_i^{(r)} & (j \neq i), \end{cases} \\ \dot{T}_i(f_j) &= \begin{cases} -k_i^{-1} e_i & (j = i) \\ \sum_{r=0}^{-a_{ij}} (-1)^{-a_{ij}-r} q_i^{-a_{ij}-r} f_i^{(-a_{ij}-r)} f_j f_i^{(r)} & (j \neq i), \end{cases} \\ \hat{T}_i(e_j) &= \begin{cases} -f_i k_i^{-1} & (j = i) \\ \sum_{r=0}^{-a_{ij}} (-1)^r q_i^r e_i^{(-a_{ij}-r)} e_j e_i^{(r)} & (j \neq i), \end{cases} \\ \hat{T}_i(f_j) &= \begin{cases} -k_i e_i & (j = i) \\ \sum_{r=0}^{-a_{ij}} (-1)^{-a_{ij}-r} q_i^{-(-a_{ij}-r)} f_i^{(-a_{ij}-r)} f_j f_i^{(r)} & (j \neq i), \end{cases} \\ \dot{T}_i(k_\gamma) &= \hat{T}_i(k_\gamma) = k_{s_i \gamma}. \end{aligned}$$

By [10] both $\{\dot{T}_i\}_{i \in I}$ and $\{\hat{T}_i\}_{i \in I}$ satisfy the braid relation, and hence we obtain the operators $\{\dot{T}_w\}_{w \in W}, \{\hat{T}_w\}_{w \in W}$ given by

$$\dot{T}_w = \dot{T}_{i_1} \cdots \dot{T}_{i_{\ell(w)}}, \quad \hat{T}_w = \hat{T}_{i_1} \cdots \hat{T}_{i_{\ell(w)}} \quad (i_1, \dots, i_{\ell(w)} \in \mathcal{I}_w).$$

By the description of \dot{T}_i, \hat{T}_i as automorphisms of U we have

$$(2.11) \quad \varepsilon(\dot{T}_w(u)) = \varepsilon(\hat{T}_w(u)) = \varepsilon(u) \quad (w \in W, u \in U).$$

For $w \in W$ and $M \in \text{Mod}_0^r(U)$ we define a right action of \dot{T}_w (resp. \hat{T}_w) on M by

$$\langle m \dot{T}_w, m^* \rangle = \langle m, \dot{T}_w m^* \rangle \quad (\text{resp. } \langle m \hat{T}_w, m^* \rangle = \langle m, \hat{T}_w m^* \rangle)$$

for $m \in M, m^* \in M^*$. We can easily check the following fact.

LEMMA 2.4. *Let $w \in W$. Then as algebra automorphisms of U we have $\hat{T}_w = S^{-1} \dot{T}_w S$.*

Let $w \in W$ and $\mathbf{i} = (i_1, \dots, i_m) \in \mathcal{I}_w$. For $r = 1, \dots, m$ set

$$\begin{aligned} k_{\mathbf{i},r} &= k_{s_{i_1} \cdots s_{i_{r-1}} \alpha_{i_r}}, \\ \dot{e}_{\mathbf{i},r} &= \dot{T}_{i_1} \cdots \dot{T}_{i_{r-1}}(e_{i_r}), & \dot{f}_{\mathbf{i},r} &= \dot{T}_{i_1} \cdots \dot{T}_{i_{r-1}}(f_{i_r}), \\ \tilde{e}_{\mathbf{i},r} &= \dot{T}_{i_m}^{-1} \cdots \dot{T}_{i_{r+1}}^{-1}(e_{i_r}), & \tilde{f}_{\mathbf{i},r} &= \dot{T}_{i_m}^{-1} \cdots \dot{T}_{i_{r+1}}^{-1}(f_{i_r}), \\ \hat{e}_{\mathbf{i},r} &= \hat{T}_{i_1} \cdots \hat{T}_{i_{r-1}}(e_{i_r}), & \hat{f}_{\mathbf{i},r} &= \hat{T}_{i_1} \cdots \hat{T}_{i_{r-1}}(f_{i_r}). \end{aligned}$$

By [10] we have $\dot{e}_{\mathbf{i},r}, \tilde{e}_{\mathbf{i},r}, \hat{e}_{\mathbf{i},r} \in U^+$, $\dot{f}_{\mathbf{i},r}, \tilde{f}_{\mathbf{i},r}, \hat{f}_{\mathbf{i},r} \in U^-$. For $n \in \mathbb{Z}_{\geq 0}$ set

$$\begin{aligned}\dot{e}_{\mathbf{i},r}^{(n)} &= \dot{T}_{i_1} \cdots \dot{T}_{i_{r-1}}(e_{i_r}^{(n)}), & \tilde{f}_{\mathbf{i},r}^{(n)} &= \dot{T}_{i_m}^{-1} \cdots \dot{T}_{i_{r+1}}^{-1}(f_{i_r}^{(n)}), \\ \hat{e}_{\mathbf{i},r}^{(n)} &= \hat{T}_{i_1} \cdots \hat{T}_{i_{r-1}}(e_{i_r}^{(n)}),\end{aligned}$$

and for $\mathbf{n} = (n_1, \dots, n_m) \in (\mathbb{Z}_{\geq 0})^m$ set

$$\begin{aligned}\dot{e}_{\mathbf{i}}^{(\mathbf{n})} &= \dot{e}_{\mathbf{i},m}^{(n_m)} \cdots \dot{e}_{\mathbf{i},1}^{(n_1)}, & \dot{f}_{\mathbf{i}}^{\mathbf{n}} &= \dot{f}_{\mathbf{i},m}^{n_m} \cdots \dot{f}_{\mathbf{i},1}^{n_1}, \\ \tilde{e}_{\mathbf{i}}^{\mathbf{n}} &= \tilde{e}_{\mathbf{i},1}^{n_1} \cdots \tilde{e}_{\mathbf{i},m}^{n_m}, & \tilde{f}_{\mathbf{i}}^{(\mathbf{n})} &= \tilde{f}_{\mathbf{i},1}^{(n_1)} \cdots \tilde{f}_{\mathbf{i},m}^{(n_m)}, \\ \hat{e}_{\mathbf{i}}^{(\mathbf{n})} &= \hat{e}_{\mathbf{i},m}^{(n_m)} \cdots \hat{e}_{\mathbf{i},1}^{(n_1)}, & \hat{f}_{\mathbf{i}}^{\mathbf{n}} &= \hat{f}_{\mathbf{i},m}^{n_m} \cdots \hat{f}_{\mathbf{i},1}^{n_1}.\end{aligned}$$

PROPOSITION 2.5 ([9]). *Let $w \in W$ and $\mathbf{i} \in \mathcal{I}_w$. Then we have*

$$\tau(\dot{e}_{\mathbf{i}}^{(\mathbf{n})}, \hat{f}_{\mathbf{i}}^{\mathbf{n}'}) = \delta_{\mathbf{n}, \mathbf{n}'} \prod_{r=1}^{\ell(w)} c_{q_{i_r}}(n_r),$$

where

$$c_q(n) = [n]! q^{-n(n-1)/2} (q - q^{-1})^{-n}.$$

The following result will be used frequently in this paper.

PROPOSITION 2.6 ([7], [9], [10]). *We have*

$$\begin{aligned}\Delta(\dot{T}_i) &= (\dot{T}_i \otimes \dot{T}_i) \exp_{q_i}((q_i - q_i^{-1})f_i \otimes e_i) \\ &= \exp_{q_i}((q_i - q_i^{-1})k_i^{-1}e_i \otimes f_i k_i)(\dot{T}_i \otimes \dot{T}_i),\end{aligned}$$

where

$$\exp_q(x) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{[n]!} x^n.$$

COROLLARY 2.7. *For $w \in W$ and $\mathbf{i} = (i_1, \dots, i_m) \in \mathcal{I}_w$ we have*

$$\begin{aligned}\Delta(\dot{T}_w) &= (\dot{T}_w \otimes \dot{T}_w) \exp_{q_{i_1}}(X_1) \cdots \exp_{q_{i_m}}(X_m) \\ &= \exp_{q_{i_1}}(Y_1) \cdots \exp_{q_{i_m}}(Y_m)(\dot{T}_w \otimes \dot{T}_w), \\ \Delta(\dot{T}_w^{-1}) &= \exp_{q_{i_m}^{-1}}(-X_m) \cdots \exp_{q_{i_1}^{-1}}(-X_1)(\dot{T}_w^{-1} \otimes \dot{T}_w^{-1}) \\ &= (\dot{T}_w^{-1} \otimes \dot{T}_w^{-1}) \exp_{q_{i_m}^{-1}}(-Y_m) \cdots \exp_{q_{i_1}^{-1}}(-Y_1),\end{aligned}$$

where

$$X_r = (q_{i_r} - q_{i_r}^{-1})\tilde{f}_{\mathbf{i},r} \otimes \tilde{e}_{\mathbf{i},r}, \quad Y_r = (q_{i_r} - q_{i_r}^{-1})k_{\mathbf{i},r}^{-1}\dot{e}_{\mathbf{i},r} \otimes \dot{f}_{\mathbf{i},r}k_{\mathbf{i},r}.$$

LEMMA 2.8. *For $w \in W$ we have*

$$\begin{aligned}\Delta(\dot{T}_w(U^+)) &\subset U \otimes (\dot{T}_w(U^+))U^0, & \Delta(\dot{T}_w^{-1}(U^+)) &\subset (\dot{T}_w^{-1}(U^+))U^0 \otimes U, \\ \Delta(\dot{T}_w(U^-)) &\subset (\dot{T}_w(U^-))U^0 \otimes U, & \Delta(\dot{T}_w^{-1}(U^+)) &\subset U \otimes (\dot{T}_w^{-1}(U^+))U^0.\end{aligned}$$

PROOF. We only show the first formula since the proof of other formulas are similar. To show the first formula we need to show that for $y \in \dot{T}_w U^+$ we have $(\dot{T}_w^{-1} \otimes \dot{T}_w^{-1})(\Delta(y)) \in U \otimes U^{\geq 0}$.

For each $\lambda \in P^-$ take $v_{w_0\lambda} \in V(\lambda)_{w_0\lambda} \setminus \{0\}$. Then for $u \in U$ we have $u \in U^{\geq 0}$ if and only if $u(M \otimes v_{w_0\lambda}) \subset M \otimes v_{w_0\lambda}$ for any $\lambda \in P^-$ and

$M \in \text{Mod}_0(U)$ (see the proof of [3, Proposition 5.11]). By this fact together with [3, Proposition 5.11] it is sufficient to show that for $M_1, M_2 \in \text{Mod}_0(U)$ and $\lambda \in P^-$ the element

$$(\text{id} \otimes \Delta)\{(\dot{T}_w^{-1} \otimes \dot{T}_w^{-1})(\Delta(y))\} \in U \otimes U \otimes U$$

sends $M_1 \otimes M_2 \otimes v_{w_0\lambda}$ to itself. As an operator on the tensor product of two integrable modules we have

$$(\dot{T}_w^{-1} \otimes \dot{T}_w^{-1})(\Delta(y)) = (\dot{T}_w^{-1} \otimes \dot{T}_w^{-1}) \circ (\Delta(y)) \circ (\dot{T}_w \otimes \dot{T}_w).$$

Take $\mathbf{i} = (i_1, \dots, i_m) \in \mathcal{I}_w$. By Corollary 2.7 we have

$$\dot{T}_w \otimes \dot{T}_w = \Delta(\dot{T}_w) \circ Z^{-1}, \quad Z = \exp_{q_{i_1}}(X_1) \cdots \exp_{q_{i_m}}(X_m),$$

where X_1, \dots, X_m are as in Corollary 2.7. Hence we have

$$(\dot{T}_w^{-1} \otimes \dot{T}_w^{-1}) \circ (\Delta(y)) \circ (\dot{T}_w \otimes \dot{T}_w) = Z \circ \Delta(\dot{T}_w^{-1}(y)) \circ Z^{-1}.$$

Therefore, our assertion is a consequence of $\dot{T}_w^{-1}(y) \in U^+$, and $X_r \in U^- \otimes U^+$. \square

For $w \in W$ set

$$(2.12) \quad U^-[\dot{T}_w] = U^- \cap \dot{T}_w(U^{\geq 0}), \quad U^+[\dot{T}_w] = U^+ \cap \dot{T}_w(U^{\leq 0}),$$

$$(2.13) \quad U^-[\dot{T}_w^{-1}] = U^- \cap \dot{T}_w^{-1}(U^{\geq 0}), \quad U^+[\dot{T}_w^{-1}] = U^+ \cap \dot{T}_w^{-1}(U^{\leq 0}),$$

$$(2.14) \quad U^-[\hat{T}_w] = U^- \cap \hat{T}_w(U^{\geq 0}), \quad U^+[\hat{T}_w] = U^+ \cap \hat{T}_w(U^{\leq 0}).$$

We can easily show the following using Lemma 2.4.

LEMMA 2.9. *For $w \in W$, $\mathbf{i} = (i_1, \dots, i_m) \in \mathcal{I}_w$ we have*

$$S^{-1}\dot{T}_w(\tilde{e}_{\mathbf{i}}^n) = \hat{f}_{\mathbf{i}}^n.$$

Hence we have

$$\dot{T}_w^{-1}S(U^-[\hat{T}_w]) = U^+[\dot{T}_w^{-1}].$$

The following well-known result is an easy consequence of the existence of the PBW-type base of U^{\pm} . Here, we give another proof which works for the quantized enveloping algebra of any symmetrizable Kac-Moody Lie algebra.

PROPOSITION 2.10. *The multiplication of U induces the isomorphisms*

$$(2.15) \quad U^{\pm} \cong U^{\pm}[\dot{T}_w] \otimes (U^{\pm} \cap \dot{T}_w(U^{\pm}))$$

$$\cong (U^{\pm} \cap \dot{T}_w(U^{\pm})) \otimes U^{\pm}[\dot{T}_w],$$

$$(2.16) \quad U^{\pm} \cong U^{\pm}[\dot{T}_w^{-1}] \otimes (U^{\pm} \cap \dot{T}_w^{-1}(U^{\pm}))$$

$$\cong (U^{\pm} \cap \dot{T}_w^{-1}(U^{\pm})) \otimes U^{\pm}[\dot{T}_w^{-1}],$$

$$(2.17) \quad U^{\pm} \cong U^{\pm}[\hat{T}_w] \otimes (U^{\pm} \cap \hat{T}_w(U^{\pm}))$$

$$\cong (U^{\pm} \cap \hat{T}_w(U^{\pm})) \otimes U^{\pm}[\hat{T}_w].$$

PROOF. We first note that (2.17) follows easily from (2.15) and Lemma 2.4. Consider the ring involution

$$a : U \rightarrow U \quad (q \mapsto q^{-1}, k_{\lambda} \mapsto k_{\lambda}^{-1}, e_i \mapsto -k_i^{-1}e_i, f_i \mapsto -f_i k_i),$$

and the ring anti-involution

$$b : U \rightarrow U \quad (q \mapsto q^{-1}, k_{\lambda} \mapsto k_{\lambda}^{-1}, e_i \mapsto f_i, f_i \mapsto e_i).$$

By $b\dot{T}_w = \dot{T}_wb$ and $b(U^+) = U^-$ the statements for U^- are consequences of those for U^+ . By $a\dot{T}_wa = \dot{T}_w^{-1}$ and $a(U_\gamma^+) = k_\gamma^{-1}U_\gamma^+$ ($\gamma \in Q^+$) the statements for \dot{T}_w^{-1} are consequences of those for \dot{T}_w . Hence we have only to show

$$(2.18) \quad U^+ \cong U^+[\dot{T}_w] \otimes (U^+ \cap \dot{T}_w(U^+)),$$

$$(2.19) \quad U^+ \cong (U^+ \cap \dot{T}_w(U^+)) \otimes U^+[\dot{T}_w].$$

We first show (2.18). Note that the injectivity of

$$(2.20) \quad U^+[\dot{T}_w] \otimes (U^+ \cap \dot{T}_w(U^+)) \rightarrow U^+$$

is clear from

$$U^+[\dot{T}_w] \otimes (U^+ \cap \dot{T}_w(U^+)) \subset T_w(U^{\leq 0}) \otimes T_w(U^+)$$

and $U^{\leq 0} \otimes U^+ \cong U$. Hence we have only to show the surjectivity of (2.20).

For $\mathbf{i} = (i_1, \dots, i_m) \in \mathcal{I}_w$ denote by $U^+[\dot{T}_w; \mathbf{i}]$ the subalgebra of U^+ generated by $\dot{e}_{i_1}, \dots, \dot{e}_{i_m}$. By a standard property of \dot{T}_i we have $U^+[\dot{T}_w; \mathbf{i}] \subset U^+[\dot{T}_w]$. Hence we have only to show

$$(2.21) \quad U^+[\dot{T}_w; \mathbf{i}](U^+ \cap \dot{T}_w(U^+)) = U^+.$$

We note that our assertion is already known for $w = s_i$. Namely, we have

$$(2.22) \quad U^+ \cong U^+[\dot{T}_i] \otimes (U^+ \cap \dot{T}_i(U^+)), \quad U^+[\dot{T}_i] = \mathbb{F}[e_i],$$

$$(2.23) \quad U^- \cong U^-[\dot{T}_i] \otimes (U^- \cap \dot{T}_i(U^-)), \quad U^-[\dot{T}_i] = \mathbb{F}[f_i]$$

(see [10, Chapter 38]).

Now we are going to show (2.21) by induction on $\ell(w)$. Assume that we have $xs_i > x$ for $x \in W$ and $i \in I$. By the above argument we need to show (2.21) for $w = xs_i$ assuming (2.15), (2.16), (2.21) for $w \in W$ with $\ell(w) \leq \ell(x)$. Take $\mathbf{i} = (i_1, \dots, i_m) \in \mathcal{I}_x$, and set $\mathbf{i}' = (i_1, \dots, i_m, i) \in \mathcal{I}_{xs_i}$. To show our assertion $U^+[\dot{T}_{xs_i}; \mathbf{i}'](U^+ \cap \dot{T}_{xs_i}(U^+)) = U^+$, it is sufficient to show

$$(2.24) \quad U^+ \cap \dot{T}_x^{-1}(U^+) = \mathbb{F}[e_i](U^+ \cap \dot{T}_i(U^+) \cap \dot{T}_x^{-1}(U^+)).$$

Indeed assuming (2.24) we have

$$\begin{aligned} U^+ \cap \dot{T}_x(U^+) &= \dot{T}_x(U^+ \cap \dot{T}_x^{-1}(U^+)) = \mathbb{F}[\dot{T}_x(e_i)](U^+ \cap \dot{T}_x(U^+) \cap \dot{T}_{xs_i}(U^+)) \\ &\subset \mathbb{F}[\dot{T}_x(e_i)](U^+ \cap \dot{T}_{xs_i}(U^+)), \end{aligned}$$

and hence

$$\begin{aligned} U^+ &= U^+[\dot{T}_x; \mathbf{i}](U^+ \cap \dot{T}_x(U^+)) \subset U^+[\dot{T}_x; \mathbf{i}]\mathbb{F}[\dot{T}_x(e_i)](U^+ \cap \dot{T}_{xs_i}(U^+)) \\ &\subset U^+[\dot{T}_{xs_i}; \mathbf{i}'](U^+ \cap \dot{T}_{xs_i}(U^+)). \end{aligned}$$

To verify (2.24) we first show the following.

$$(2.25) \quad \begin{aligned} &U^+ \cap \dot{T}_x^{-1}(U^+) \\ &= \{u \in U^+ \mid \tau(u, U^-(U^+[\dot{T}_x^{-1}] \cap \text{Ker}(\varepsilon : U^- \rightarrow \mathbb{F}))) = \{0\}\}, \end{aligned}$$

$$(2.26) \quad \begin{aligned} &U^- \cap \dot{T}_x^{-1}(U^-) \\ &= \{u \in U^- \mid \tau(U^+(U^+[\dot{T}_x^{-1}] \cap \text{Ker}(\varepsilon : U^+ \rightarrow \mathbb{F})), u) = \{0\}\}. \end{aligned}$$

For simplicity set

$$\begin{aligned} V^+ &= \{u \in U^+ \mid \tau(u, U^-(U^-[\dot{T}_x^{-1}] \cap \text{Ker}(\varepsilon))) = \{0\}\}, \\ V^- &= \{u \in U^- \mid \tau(U^+(U^+[\dot{T}_x^{-1}] \cap \text{Ker}(\varepsilon)), u) = \{0\}\}. \end{aligned}$$

By (2.9) we have

$$\tau(U^+ \cap \dot{T}_x^{-1}(U^+), U^-[\dot{T}_x^{-1}] \cap \text{Ker}(\varepsilon)) = \{0\}.$$

Hence by Lemma 2.8 and (2.2) We have $U^+ \cap \dot{T}_x^{-1}(U^+) \subset V^+$. By a similar argument we have also $U^- \cap \dot{T}_x^{-1}(U^-) \subset V^-$. On the other hand by the hypothesis of induction we have $U^\pm \cong U^\pm[\dot{T}_x^{-1}] \otimes (U^\pm \cap \dot{T}_x^{-1}(U^\pm))$, and hence $U^\pm = (U^\pm \cap \dot{T}_x^{-1}(U^\pm)) \oplus U^\pm(U^\pm[\dot{T}_x^{-1}] \cap \text{Ker}(\varepsilon))$. Since τ is non-degenerate, we obtain injective linear maps $V^\pm \rightarrow (U^\mp \cap \dot{T}_x^{-1}(U^\mp))^*$. Comparing the dimensions of weight spaces we obtain

$$\dim(U_{\pm\gamma}^\pm \cap \dot{T}_x^{-1}(U^\pm)) \leq \dim V_{\pm\gamma}^\pm \leq \dim(U_{\mp\gamma}^\mp \cap \dot{T}_x^{-1}(U^\mp)).$$

for each $\gamma \in Q^+$. This gives (2.25) and (2.26).

By a similar argument we have also

$$\begin{aligned} (2.27) \quad & U^+ \cap \dot{T}_x(U^+) \\ &= \{u \in U^+ \mid \tau(u, (U^-[\dot{T}_x] \cap \text{Ker}(\varepsilon : U^- \rightarrow \mathbb{F}))U^-) = \{0\}\}, \end{aligned}$$

$$\begin{aligned} (2.28) \quad & U^- \cap \dot{T}_x(U^-) \\ &= \{u \in U^- \mid \tau((U^+[\dot{T}_x] \cap \text{Ker}(\varepsilon : U^+ \rightarrow \mathbb{F}))U^+, u) = \{0\}\}. \end{aligned}$$

Now let us show (2.24). Take $u \in U^+ \cap \dot{T}_x^{-1}(U^+)$, and decompose it as

$$u = \sum_n e_i^n u_n \quad (u_n \in U^+ \cap \dot{T}_i(U^+))$$

(see (2.22)). Then it is sufficient to show $u_n \in \dot{T}_x^{-1}(U^+)$, which is equivalent to

$$(2.29) \quad \tau(u_n, vz) = 0 \quad (v \in U^-, z \in U^-[\dot{T}_x^{-1}] \cap \text{Ker}(\varepsilon))$$

by (2.25). Let $v \in U^-$, $z \in U^-[\dot{T}_x^{-1}] \cap \text{Ker}(\varepsilon)$. By our assumption we have $\tau(u, vz) = 0$. On the other hand we have

$$\begin{aligned} \tau(u, vz) &= \sum_n \tau(e_i^n u_n, vz) = \sum_n \sum_{(z), (v)} \tau(e_i^n, v_{(0)} z_{(0)}) \tau(u_n, v_{(1)} z_{(1)}) \\ &= \sum_n \sum_{(v)} \tau(e_i^n, v_{(0)}) \tau(u_n, v_{(1)} z) \end{aligned}$$

by Lemma 2.8 and (2.9). Consider the case

$$v = f_i^{(r)} v' \quad (v' \in U^- \cap \dot{T}_i(U^-)).$$

Then we have

$$\begin{aligned} \tau(u, vz) &= \sum_n \sum_{s=0}^r \sum_{(v')} q_i^{-s(r-s)} \tau(e_i^n, f_i^{(r-s)} v'_{(0)}) \tau(u_n, k_i^{-(r-s)} f_i^{(s)} v'_{(1)} z) \\ &= \sum_n \sum_{s=0}^r q_i^{-s(r-s)} \tau(e_i^n, f_i^{(r-s)}) \tau(u_n, f_i^{(s)} v' z) \end{aligned}$$

by Lemma 2.8, (2.8), (2.9). By $u_n \in U^+ \cap \dot{T}_i(U^+)$ and (2.27) we have

$$(2.30) \quad \tau(u_n, f_i U^-) = \{0\}.$$

Hence

$$\tau(u, vz) = \sum_n \tau(e_i^n, f_i^{(r)}) \tau(u_n, v'z) = \tau(e_i^r, f_i^{(r)}) \tau(u_r, v'z).$$

Hence by $\tau(e_i^r, f_i^{(r)}) \neq 0$ we obtain

$$(2.31) \quad \tau(u_n, v'z) = 0 \quad (z \in U^-[\dot{T}_x^{-1}] \cap \text{Ker}(\varepsilon), v' \in U^- \cap \dot{T}_i(U^-))$$

for any n . By (2.30), (2.31)

$$\tau(u_n, f_i^r v'z) = 0 \quad (z \in U^-[\dot{T}_x^{-1}] \cap \text{Ker}(\varepsilon), v' \in U^- \cap \dot{T}_i(U^-), r \geq 0).$$

The proof of (2.18) is complete.

It remains to show (2.19). Similarly to the above proof of (2.18) it is sufficient to show

$$(2.32) \quad U^+ \cap \dot{T}_x^{-1}(U^+) = (U^+ \cap \dot{T}_i(U^+) \cap \dot{T}_x^{-1}(U^+))\mathbb{F}[e_i].$$

This follows from (2.24) as follows. We can easily show

$$(2.33) \quad \gamma \in Q^+, u \in U^+ \cap \dot{T}_i(U^+) \implies ue_i - q_i^{(\gamma, \alpha_i^\vee)} e_i u \in U^+ \cap \dot{T}_i(U^+)$$

using [10, Proposition 38.1.6]. Hence in view of (2.24) it is sufficient to show that for $u \in U^+ \cap \dot{T}_i(U^+) \cap \dot{T}_x^{-1}(U^+)$ we have $ue_i - q_i^{(\gamma, \alpha_i^\vee)} e_i u \in \dot{T}_x^{-1}(U^+)$. This is obvious since $\dot{T}_x(e_i) \in U^+$. \square

REMARK 2.11. Proposition 2.10 holds true for various $\mathbb{Z}[q, q^{-1}]$ -forms of U^\pm . To show this it is sufficient to verify (2.22) over $\mathbb{Z}[q, q^{-1}]$. In the case of the De Concini-Kac form $U_{\mathbb{Z}[q, q^{-1}]}^{DK, \pm}$, this follows if we can show that $(U_{\mathbb{Z}[q, q^{-1}]}^{DK, +} \cap U^+[\dot{T}_i])(U_{\mathbb{Z}[q, q^{-1}]}^{DK, +} \cap \dot{T}_i(U^+))$ is stable under the right multiplication of e_j for any $j \in I$. If $j \neq i$, this is obvious. If $j = i$, this follows from (2.33). The argument for the case of the Lusztig form $U_{\mathbb{Z}[q, q^{-1}]}^{L, \pm}$ is similar. Finally, (2.22) over $\mathbb{Z}[q, q^{-1}]$ for the De Concini-Procesi form defined by

$$\begin{aligned} U_{\mathbb{Z}[q, q^{-1}]}^{DP, +} &= \{u \in U^+ \mid \tau(u, U_{\mathbb{Z}[q, q^{-1}]}^{L, -}) \in \mathbb{Z}[q, q^{-1}]\}, \\ U_{\mathbb{Z}[q, q^{-1}]}^{DP, -} &= \{u \in U^- \mid \tau(U_{\mathbb{Z}[q, q^{-1}]}^{L, +}, u) \in \mathbb{Z}[q, q^{-1}]\}, \end{aligned}$$

is a consequence of that for the Lusztig form by duality.

By our proof of Proposition 2.10 we also obtain the following.

PROPOSITION 2.12. *Let $w \in W$ and $\mathbf{i} \in \mathcal{I}_w$.*

- (i) *The set $\{\dot{f}_i^{\mathbf{n}} \mid \mathbf{n} \in (\mathbb{Z}_{\geq 0})^m\}$ (resp. $\{\dot{e}_i^{(\mathbf{n})} \mid \mathbf{n} \in (\mathbb{Z}_{\geq 0})^m\}$) forms an \mathbb{F} -basis of $U^-[\dot{T}_w]$ (resp. $U^+[\dot{T}_w]$).*
- (ii) *The set $\{\tilde{f}_i^{\mathbf{n}} \mid \mathbf{n} \in (\mathbb{Z}_{\geq 0})^m\}$ (resp. $\{\tilde{e}_i^{\mathbf{n}} \mid \mathbf{n} \in (\mathbb{Z}_{\geq 0})^m\}$) forms an \mathbb{F} -basis of $U^-[\dot{T}_w^{-1}]$ (resp. $U^+[\dot{T}_w^{-1}]$).*
- (iii) *The set $\{\hat{f}_i^{\mathbf{n}} \mid \mathbf{n} \in (\mathbb{Z}_{\geq 0})^m\}$ (resp. $\{\hat{e}_i^{(\mathbf{n})} \mid \mathbf{n} \in (\mathbb{Z}_{\geq 0})^m\}$) forms an \mathbb{F} -basis of $U^-[\hat{T}_w]$ (resp. $U^+[\hat{T}_w]$).*

For $w \in W$ and $\gamma \in Q^+$ we have

$$(2.34) \quad U^\pm[\dot{T}_w] \cap U_{\pm\gamma}^\pm = U^\pm \cap \dot{T}_w(k_{\pm w^{-1}\gamma} U_{\pm w^{-1}\gamma}^\mp),$$

$$(2.35) \quad U^\pm[\dot{T}_w^{-1}] \cap U_{\pm\gamma}^\pm = U^\pm \cap \dot{T}_w^{-1}(k_{\mp w\gamma} U_{\pm w\gamma}^\mp)$$

by Proposition 2.12 and the explicit description of \dot{T}_i .

3. QUANTIZED COORDINATE ALGEBRAS

3.1. We denote by $\mathbb{C}_q[G]$ the quantized coordinate algebra of U (see, for example, [4], [5], [13] for the basic facts concerning $\mathbb{C}_q[G]$). It is the \mathbb{F} -subspace of U^* spanned by the matrix coefficients of U -modules belonging to $\text{Mod}_0(U)$. Namely, for $V \in \text{Mod}_0(U)$ define a linear map

$$(3.1) \quad \Phi : V^* \otimes V \rightarrow U^* \quad (v^* \otimes v \mapsto \Phi_{v^* \otimes v})$$

by

$$\langle \Phi_{v^* \otimes v}, u \rangle = \langle v^*, uv \rangle \quad (u \in U).$$

Then we have

$$(3.2) \quad \mathbb{C}_q[G] = \sum_{V \in \text{Mod}_0(U)} \text{Im}(V^* \otimes V \rightarrow U^*) \subset U^*.$$

It is endowed with a Hopf algebra structure by $m_{\mathbb{C}_q[G]} = {}^t\Delta_U$, $\Delta_{\mathbb{C}_q[G]} = {}^t m_U$. Moreover, (3.2) induces

$$(3.3) \quad \mathbb{C}_q[G] \cong \bigoplus_{\lambda \in P^-} V^*(\lambda) \otimes V(\lambda).$$

We set

$$\begin{aligned} \mathbb{C}_q[B^+] &= \text{Im}(\mathbb{C}_q[G] \rightarrow (U^{\geq 0})^*), \quad \mathbb{C}_q[B^-] = \text{Im}(\mathbb{C}_q[G] \rightarrow (U^{\leq 0})^*), \\ \mathbb{C}_q[H] &= \text{Im}(\mathbb{C}_q[G] \rightarrow (U^0)^*). \end{aligned}$$

For $\lambda \in P$ we denote by $\chi_\lambda : U^0 \rightarrow \mathbb{F}$ the algebra homomorphism given by $\chi_\lambda(k_\gamma) = q^{(\lambda, \gamma)}$ for $\gamma \in Q$. Then we have

$$(3.4) \quad \mathbb{C}_q[H] = \bigoplus_{\lambda \in P} \mathbb{F}\chi_\lambda.$$

Note that $\mathbb{C}_q[G]$ is a U -bimodule by

$$\langle u_1 \varphi u_2, u \rangle = \langle \varphi, u_2 u u_1 \rangle \quad (\varphi \in \mathbb{C}_q[G], u_1, u_2, u \in U).$$

For $\varphi, \psi \in \mathbb{C}_q[G]$ and $u \in U$ we have

$$(3.5) \quad u(\varphi\psi) = \sum_{(u)} (u_{(0)}\varphi)(u_{(1)}\psi),$$

$$(3.6) \quad (\varphi\psi)u = \sum_{(u)} (\varphi u_{(0)})(\psi u_{(1)}).$$

EXAMPLE 3.1. Consider the case where $\mathfrak{g} = \mathfrak{sl}_2$ and $G = SL_2$. In this case $U = U_q(\mathfrak{sl}_2)$ is the \mathbb{F} -algebra generated by the elements $k^{\pm 1}$, e , f satisfying

$$kk^{-1} = k^{-1}k = 1, \quad kek^{-1} = q^2e, \quad kfk^{-1} = q^{-2}f, \quad ef - fe = \frac{k - k^{-1}}{q - q^{-1}}.$$

Let $V = \mathbb{F}v_0 \oplus \mathbb{F}v_1$ be the two-dimensional U -module given by

$$kv_0 = qv_0, \quad kv_1 = q^{-1}v_1, \quad ev_0 = 0, \quad ev_1 = v_0, \quad fv_0 = v_1, \quad fv_1 = 0,$$

and define $a, b, c, d \in \mathbb{C}_q[SL_2]$ by

$$uv_0 = \langle a, u \rangle v_0 + \langle c, u \rangle v_1, \quad uv_1 = \langle b, u \rangle v_0 + \langle d, u \rangle v_1 \quad (u \in U).$$

Then $\{a, b, c, d\}$ forms a generator system of the \mathbb{F} -algebra $\mathbb{C}_q[SL_2]$ satisfying the fundamental relations

$$\begin{aligned} ab &= qba, & cd &= qdc, & ac &= qca, & bd &= qdb, & bc &= cb, \\ 1ad - da &= (q - q^{-1})bc, & ad - qbc &= 1. \end{aligned}$$

Its Hopf algebra structure is given by

$$\begin{aligned} \Delta(a) &= a \otimes a + b \otimes c, & \Delta(b) &= a \otimes b + b \otimes d, \\ \Delta(c) &= c \otimes a + d \otimes c, & \Delta(d) &= c \otimes b + d \otimes d, \\ \varepsilon(a) &= \varepsilon(d) = 1, & \varepsilon(b) &= \varepsilon(c) = 0, \\ S(a) &= d, & S(b) &= -q^{-1}b, & S(c) &= -qc, & S(d) &= a. \end{aligned}$$

3.2. For $w \in W$ and $\lambda \in P^-$ set

$$v_{w\lambda}^* = v_\lambda^* \dot{T}_w^{-1} \in V^*(\lambda)_{w\lambda},$$

and define $\sigma_\lambda^w \in \mathbb{C}_q[G]$ by

$$\langle \sigma_\lambda^w, u \rangle = \langle v_{w\lambda}^*, uv_\lambda \rangle \quad (u \in U).$$

Then we have

$$(3.7) \quad \sigma_0^w = 1, \quad \sigma_\lambda^w \sigma_\mu^w = \sigma_\mu^w \sigma_\lambda^w = \sigma_{\lambda+\mu}^w \quad (\lambda, \mu \in P^-).$$

Set

$$\mathcal{S}_w = \{\sigma_\lambda^w \mid \lambda \in P^-\} \subset \mathbb{C}_q[G].$$

PROPOSITION 3.2 ([4]). *The multiplicative subset \mathcal{S}_w of $\mathbb{C}_q[G]$ satisfies the left and right Ore conditions.*

It follows that we have the localization $\mathcal{S}_w^{-1}\mathbb{C}_q[G] = \mathbb{C}_q[G]\mathcal{S}_w^{-1}$. In the rest of this section we investigate the structure of the algebra $\mathcal{S}_w^{-1}\mathbb{C}_q[G]$. In the course of the arguments we give a new proof of Proposition 3.2.

3.3. In this subsection we consider the case $w = 1$.

Set

$$(U^\pm)^\star = \bigoplus_{\gamma \in Q^+} (U_{\pm\gamma}^\pm)^* \subset U^*.$$

LEMMA 3.3. *We have*

$$\mathbb{C}_q[G] \subset (U^+)^\star \otimes \mathbb{C}_q[H] \otimes (U^-)^\star \subset (U^+)^* \otimes (U^0)^* \otimes (U^-)^* \subset U^*,$$

where the embedding $(U^+)^* \otimes (U^0)^* \otimes (U^-)^* \subset U^*$ is given by

$$\begin{aligned} \langle \psi \otimes \chi \otimes \varphi, xty \rangle &= \langle \psi, x \rangle \langle \chi, t \rangle \langle \varphi, y \rangle \\ (\psi &\in (U^+)^*, \chi \in (U^0)^*, \varphi \in (U^-)^*, x \in U^+, t \in U^0, y \in U^-). \end{aligned}$$

PROOF. It is easily seen that for any $\varphi \in \mathbb{C}_q[G]$ we have

$$\varphi|_{U^0} \in \mathbb{C}_q[H], \quad \varphi|_{U^\pm} \in (U^\pm)^\star.$$

Hence the assertion is a consequence of

$$\langle \varphi, xty \rangle = \sum_{(\varphi)_2} \langle \varphi_{(0)}, x \rangle \langle \varphi_{(1)}, t \rangle \langle \varphi_{(2)}, y \rangle \quad (x \in U^+, t \in U^0, y \in U^-)$$

for $\varphi \in \mathbb{C}_q[G]$. \square

Note that U^* is an \mathbb{F} -algebra whose multiplication is given by the composite of $U^* \otimes U^* \subset (U \otimes U)^* \xrightarrow{t\Delta} U^*$ and that $\mathbb{C}_q[G]$ is a subalgebra of U^* . We will identify $(U^0)^*$, $(U^\pm)^*$ with subspaces of U^* by

$$\begin{aligned} (U^+)^* &\rightarrow U^* & (\psi &\mapsto [xty \mapsto \langle \psi, x \rangle \varepsilon(t) \varepsilon(y)]), \\ (U^0)^* &\rightarrow U^* & (\chi &\mapsto [xty \mapsto \varepsilon(x) \langle \chi, t \rangle \varepsilon(y)]), \\ (U^-)^* &\rightarrow U^* & (\varphi &\mapsto [xty \mapsto \varepsilon(x) \varepsilon(t) \langle \varphi, y \rangle]), \end{aligned}$$

where $x \in U^+$, $t \in U^0$, $y \in U^-$. Under this identification we have

$$(3.8) \quad \chi_\lambda = \sigma_\lambda^1 \quad (\lambda \in P^-).$$

Hence

$$(3.9) \quad \mathcal{S}_1 = \{\chi_\lambda \mid \lambda \in P^-\} \subset \mathbb{C}_q[H] \subset (U^0)^* \subset U^*.$$

LEMMA 3.4. For $\psi \in (U^+)^*$, $\chi \in (U^0)^*$, $\varphi \in (U^-)^*$ we have

$$\langle \psi \chi \varphi, xty \rangle = \langle \psi, x \rangle \langle \chi, t \rangle \langle \varphi, y \rangle \quad (y \in U^-, t \in U^0, x \in U^+).$$

PROOF. By the definition of the comultiplication of U , for $x \in U^+$, $y \in U^-$ we have

$$\begin{aligned} \Delta(y) &= 1 \otimes y + y', & (\varepsilon \otimes \text{id})(y') &= 0, \\ \Delta(x) &= x \otimes 1 + x', & (\text{id} \otimes \varepsilon)(x') &= 0. \end{aligned}$$

Hence

$$\begin{aligned} \langle \psi \chi \varphi, xty \rangle &= \sum_{(x)_2, (t)_2, (y)_2} \langle \psi, x_{(0)} t_{(0)} y_{(0)} \rangle \langle \chi, x_{(1)} t_{(1)} y_{(1)} \rangle \langle \varphi, x_{(2)} t_{(2)} y_{(2)} \rangle \\ &= \sum_{(t)_2} \langle \psi, x t_{(0)} \rangle \langle \chi, t_{(1)} \rangle \langle \varphi, t_{(2)} y \rangle = \langle \psi, x \rangle \langle \chi, t \rangle \langle \varphi, y \rangle. \end{aligned}$$

\square

LEMMA 3.5. For $\lambda, \mu \in P$ we have $\chi_\lambda \chi_\mu = \chi_{\lambda+\mu}$.

PROOF. We have

$$\begin{aligned} \langle \chi_\lambda \chi_\mu, xty \rangle &= \sum_{(x), (t), (y)} \langle \chi_\lambda, x_{(0)} t_{(0)} y_{(0)} \rangle \langle \chi_\mu, x_{(1)} t_{(1)} y_{(1)} \rangle \\ &= \varepsilon(x) \varepsilon(y) \sum_{(t)} \langle \chi_\lambda, t_{(0)} \rangle \langle \chi_\mu, t_{(1)} \rangle \\ &= \varepsilon(x) \varepsilon(y) \langle \chi_{\lambda+\mu}, t \rangle = \langle \chi_{\lambda+\mu}, xty \rangle. \end{aligned}$$

\square

LEMMA 3.6. The subspaces $(U^+)^\star$, $(U^-)^\star$ of U^* are subalgebras of U^* .

PROOF. For $\varphi, \varphi' \in (U^-)^\star$ we have

$$\begin{aligned} \langle \varphi \varphi', xty \rangle &= \sum_{(x),(t),(y)} \langle \varphi, x_{(0)} t_{(0)} y_{(0)} \rangle \langle \varphi', x_{(1)} t_{(1)} y_{(1)} \rangle \\ &= \varepsilon(x) \sum_{(t),(y)} \langle \varphi, t_{(0)} y_{(0)} \rangle \langle \varphi', t_{(1)} y_{(1)} \rangle \\ &= \varepsilon(x) \varepsilon(t) \sum_{(y)} \langle \varphi, y_{(0)} \rangle \langle \varphi', y_{(1)} \rangle = \varepsilon(x) \varepsilon(t) \langle \varphi \varphi', y \rangle. \end{aligned}$$

The statement for $(U^+)^\star$ is proved similarly. \square

LEMMA 3.7. (i) For $\psi \in (U_\gamma^+)^*$, $\lambda \in P$ we have $\chi_\lambda \psi = q^{(\lambda, \gamma)} \psi \chi_\lambda$.
(ii) For $\varphi \in (U_{-\gamma}^-)^*$, $\lambda \in P$ we have $\chi_\lambda \varphi = q^{(\lambda, \gamma)} \varphi \chi_\lambda$.

PROOF. For $x \in U_{\gamma'}^+$, $y \in U^-$, $t \in U^0$ we have

$$\begin{aligned} \langle \chi_\lambda \psi, xty \rangle &= \sum_{(x),(t),(y)} \langle \chi_\lambda, x_{(0)} t_{(0)} y_{(0)} \rangle \langle \psi, x_{(1)} t_{(1)} y_{(1)} \rangle \\ &= \sum_{(t)} \langle \chi_\lambda, k_{\gamma'} t_{(0)} \rangle \langle \psi, xt_{(1)} y \rangle = \varepsilon(t_{(1)}) \varepsilon(y) \delta_{\gamma, \gamma'} \sum_{(t)} \langle \chi_\lambda, k_{\gamma'} t_{(0)} \rangle \langle \psi, x \rangle \\ &= \varepsilon(y) \delta_{\gamma, \gamma'} \langle \chi_\lambda, k_\gamma t \rangle \langle \psi, x \rangle = q^{(\lambda, \gamma)} \varepsilon(y) \langle \psi, x \rangle \langle \chi_\lambda, t \rangle. \end{aligned}$$

By a similar calculation we have

$$\langle \psi \chi_\lambda, xty \rangle = \varepsilon(y) \langle \psi, x \rangle \langle \chi_\lambda, t \rangle.$$

The statement (i) is proved. The proof of (ii) is similar. \square

LEMMA 3.8. (i) Let $\varphi \in (U^-)^\star$. For sufficiently small $\lambda \in P^-$ we have $\chi_\lambda \varphi, \varphi \chi_\lambda \in \mathbb{C}_q[G]$.
(ii) Let $\psi \in (U^+)^\star$. For sufficiently small $\lambda \in P^-$ we have $\chi_\lambda \psi, \psi \chi_\lambda \in \mathbb{C}_q[G]$.

PROOF. (i) We may assume $\varphi \in (U_{-\gamma}^-)^*$. By Proposition 2.2 there exists $v \in V(\lambda)_{\lambda+\gamma}$ such that

$$\langle \varphi, y \rangle = \langle v_\lambda^* y, v \rangle \quad (y \in U^-).$$

Then

$$\langle \Phi_{v_\lambda^* \otimes v}, xty \rangle = \langle v_\lambda^* xty, v \rangle = \varepsilon(x) \langle \chi_\lambda, t \rangle \langle \varphi, y \rangle = \langle \chi_\lambda \varphi, xty \rangle.$$

Hence $\chi_\lambda \varphi = q^{(\lambda, \gamma)} \varphi \chi_\lambda = \Phi_{v_\lambda^* \otimes v} \in \mathbb{C}_q[G]$. The proof of (ii) is similar. \square

COROLLARY 3.9. Let $f \in (U^+)^\star \mathbb{C}_q[H] (U^-)^\star$. Then we have $\chi_\lambda f, f \chi_\lambda \in \mathbb{C}_q[G]$ for sufficiently small $\lambda \in P^-$.

PROOF. We may assume $f = \psi \chi_\nu \varphi$ ($\psi \in (U_\gamma^+)^*$, $\nu \in P$, $\varphi \in (U_{-\delta}^-)^*$). By Lemma 3.8 we have $\chi_{\lambda_1} \psi, \chi_{\lambda_3} \varphi \in \mathbb{C}_q[G]$ when $\lambda_1, \lambda_3 \in P^-$ are sufficiently small. Take $\lambda_2 \in P^-$ such that $\lambda_2 + \nu \in P^-$ and set $\lambda = \lambda_1 + \lambda_2 + \lambda_3$. Then we have

$$\chi_\lambda f = q^{(\lambda_2 + \lambda_3, \gamma)} (\chi_{\lambda_1} \psi) \chi_{\lambda_2 + \mu} (\chi_{\lambda_3} \varphi) \in \mathbb{C}_q[G].$$

The proof for $f \chi_\lambda$ is similar. \square

PROPOSITION 3.10. Let $f \in \mathbb{C}_q[G]$, $\lambda \in P^-$.

- (i) If $\sigma_\lambda^1 f = 0$, then $f = 0$.
- (ii) If $f \sigma_\lambda^1 = 0$, then $f = 0$.

PROOF. In the algebra U^* the element $\sigma_\lambda^1 = \chi_\lambda$ is invertible, and its inverse is given by $\chi_{-\lambda}$. \square

We set

$$(3.10) \quad \begin{aligned} \mathbb{C}_q[G/N^-] &= \{f \in \mathbb{C}_q[G] \mid yf = \varepsilon(y)f \ (y \in U^-)\} \\ &= \mathbb{C}_q[G] \cap (U^+)^\star \mathbb{C}_q[H], \end{aligned}$$

$$(3.11) \quad \begin{aligned} \mathbb{C}_q[N^+ \setminus G] &= \{f \in \mathbb{C}_q[G] \mid fx = \varepsilon(x)f \ (x \in U^+)\} \\ &= \mathbb{C}_q[G] \cap \mathbb{C}_q[H](U^-)^\star. \end{aligned}$$

They are subalgebras of $\mathbb{C}_q[G]$.

PROPOSITION 3.11. Assume that $\lambda \in P^-$.

- (i) $\forall \psi \in \mathbb{C}_q[G/N^-] \exists \mu \in P^-$ s.t. $\sigma_\mu^1 \psi \in \mathbb{C}_q[G/N^-] \sigma_\lambda^1$, $\psi \sigma_\mu^1 \in \sigma_\lambda^1 \mathbb{C}_q[G/N^-]$.
- (ii) $\forall \varphi \in \mathbb{C}_q[N^+ \setminus G] \exists \mu \in P^-$ s.t. $\sigma_\mu^1 \varphi \in \mathbb{C}_q[N^+ \setminus G] \sigma_\lambda^1$, $\varphi \sigma_\mu^1 \in \sigma_\lambda^1 \mathbb{C}_q[N^+ \setminus G]$.
- (iii) $\forall f \in \mathbb{C}_q[G] \exists \mu \in P^-$ s.t. $\sigma_\mu^1 f \in \mathbb{C}_q[G] \sigma_\lambda^1$, $f \sigma_\mu^1 \in \sigma_\lambda^1 \mathbb{C}_q[G]$.

PROOF. (i) By Lemma 3.7 we have

$$\sigma_\lambda^1 \psi \in \sigma_\lambda^1 (U^+)^\star \mathbb{C}_q[H] \subset (U^+)^\star \mathbb{C}_q[H] \sigma_\lambda^1.$$

By Corollary 3.9 we have $\sigma_{\lambda+\nu}^1 \psi \in \mathbb{C}_q[G/N^-] \sigma_\lambda^1$ for some $\nu \in P^-$. Similarly, we have $\psi \sigma_{\lambda+\nu'}^1 \in \sigma_\lambda^1 \mathbb{C}_q[G/N^-]$ for some $\nu' \in P^-$.

The statements (ii), (iii) are proved similarly. \square

By Proposition 3.10 and Proposition 3.11 we have the following.

COROLLARY 3.12. The multiplicative set \mathcal{S}_1 satisfies the left and right Ore conditions in all of the three rings $\mathbb{C}_q[G/N^-]$, $\mathbb{C}_q[N^+ \setminus G]$, $\mathbb{C}_q[G]$.

It follows that we have the localizations

$$(3.12) \quad \mathcal{S}_1^{-1} \mathbb{C}_q[G/N^-] = \mathbb{C}_q[G/N^-] \mathcal{S}_1^{-1},$$

$$(3.13) \quad \mathcal{S}_1^{-1} \mathbb{C}_q[N^+ \setminus G] = \mathbb{C}_q[N^+ \setminus G] \mathcal{S}_1^{-1},$$

$$(3.14) \quad \mathcal{S}_1^{-1} \mathbb{C}_q[G] = \mathbb{C}_q[G] \mathcal{S}_1^{-1}.$$

The following result is a special case of [15, Theorem 2.6].

- PROPOSITION 3.13. (i) The subset $(U^+)^\star \mathbb{C}_q[H](U^-)^\star$ of U^* is a subalgebra of U^* , which is isomorphic to $\mathcal{S}_1^{-1} \mathbb{C}_q[G]$.
- (ii) The subset $(U^+)^\star \mathbb{C}_q[H]$ of U^* is a subalgebra of U^* , which is isomorphic to $\mathcal{S}_1^{-1} \mathbb{C}_q[G/N^-]$.
- (iii) The subset $\mathbb{C}_q[H](U^-)^\star$ of U^* is a subalgebra of U^* , which is isomorphic to $\mathcal{S}_1^{-1} \mathbb{C}_q[N^+ \setminus G]$.

PROOF. (i) Since \mathcal{S}_1 consists of invertible elements of U^* , we have a canonical homomorphism $\Psi : \mathcal{S}_1^{-1} \mathbb{C}_q[G] \rightarrow U^*$ of \mathbb{F} -algebras. Since $\mathbb{C}_q[G] \rightarrow U^*$ is injective, Ψ is injective by Proposition 3.10. Hence it is sufficient to show that the image of Ψ coincides with $(U^+)^\star \mathbb{C}_q[H](U^-)^\star$. For any $\lambda \in P$ we have

$$\chi_\lambda \mathbb{C}_q[G] \subset \chi_\lambda (U^+)^\star \mathbb{C}_q[H](U^+)^\star = (U^+)^\star \mathbb{C}_q[H](U^-)^\star,$$

and hence $\text{Im}(\Psi) \subset (U^+)^\star \mathbb{C}_q[H](U^-)^\star$. Another inclusion $\text{Im}(\Psi) \supset (U^+)^\star \mathbb{C}_q[H](U^-)^\star$ is a consequence of Corollary 3.9.

The proofs of (ii) and (iii) are similar. \square

By Proposition 3.13 we obtain the following results.

PROPOSITION 3.14. *The multiplication of $\mathcal{S}_1^{-1}\mathbb{C}_q[G]$ induces the isomorphism*

$$\mathcal{S}_1^{-1}\mathbb{C}_q[G/N^-] \otimes_{\mathbb{C}_q[H]} \mathcal{S}_1^{-1}\mathbb{C}_q[N^+ \setminus G] \cong \mathcal{S}_1^{-1}\mathbb{C}_q[G].$$

PROPOSITION 3.15. *For any $f \in \mathbb{C}_q[G]$ there exists some $\lambda \in P^-$ such that*

$$\sigma_\lambda^1 f, f \sigma_\lambda^1 \in \mathbb{C}_q[G/N^-] \mathbb{C}_q[N^+ \setminus G].$$

3.4. In this subsection we investigate the localization of $\mathbb{C}_q[G]$ with respect to \mathcal{S}_w for $w \in W$.

As a left (resp. right) U -module, $\mathbb{C}_q[G]$ is a sum of submodules belonging to $\text{Mod}_0(U)$ (resp. $\text{Mod}_0^r(U)$). Hence we have a left (resp. right) action of \dot{T}_w on $\mathbb{C}_q[G]$.

LEMMA 3.16. *For $w \in W$ we have*

$$\langle \dot{T}_w \varphi, u \rangle = \langle \varphi \dot{T}_w, \dot{T}_w^{-1}(u) \rangle, \quad \langle \varphi \dot{T}_w, u \rangle = \langle \dot{T}_w \varphi, \dot{T}_w(u) \rangle.$$

for $\varphi \in \mathbb{C}_q[G]$, $u \in U$.

PROOF. We may assume that $\varphi = \Phi_{v^* \otimes v}$. Then we have

$$\langle \dot{T}_w \varphi, u \rangle = \langle v^*, u \dot{T}_w v \rangle = \langle v^* \dot{T}_w, (\dot{T}_w^{-1}(u))v \rangle = \langle \varphi \dot{T}_w, \dot{T}_w^{-1}(u) \rangle.$$

The second formula follows from the first. \square

Setting $u = 1$ in Lemma 3.16 we obtain the following.

LEMMA 3.17. *For $w \in W$ we have*

$$\varepsilon(\varphi \dot{T}_w) = \varepsilon(\dot{T}_w \varphi) \quad (\varphi \in \mathbb{C}_q[G]).$$

In the rest of this section we fix $w \in W$.

LEMMA 3.18. (i) $\sigma_\lambda^1 \dot{T}_w^{-1} = \sigma_\lambda^w$ for $\lambda \in P^-$.

(ii) $\mathbb{C}_q[G/N^-] \dot{T}_w^{-1} = \mathbb{C}_q[G/N^-]$.

(iii) $\mathbb{C}_q[N^+ \setminus G] \dot{T}_w^{-1} = \{\varphi \in \mathbb{C}_q[G] \mid \varphi u = \varepsilon(u) \varphi \text{ } (u \in \dot{T}_w(U^+))\}$.

PROOF. The statements (i) and (ii) are obvious. The remaining (iii) is a consequence of

$$(f \dot{T}_w^{-1})(\dot{T}_w(u)) = (fu) \dot{T}_w^{-1} \quad (f \in \mathbb{C}_q[G], u \in U)$$

and (2.11). \square

LEMMA 3.19. (i) $(fh) \dot{T}_w^{-1} = (f \dot{T}_w^{-1})(h \dot{T}_w^{-1})$ ($h \in \mathbb{C}_q[N^+ \setminus G]$, $f \in \mathbb{C}_q[G]$).

(ii) $(f \sigma_\lambda^1) \dot{T}_w^{-1} = (f \dot{T}_w^{-1}) \sigma_\lambda^w$ ($f \in \mathbb{C}_q[G]$).

(iii) $(\sigma_\lambda^1 f) \dot{T}_{w^{-1}} \in \mathbb{F}^\times \sigma_\lambda^w (f \dot{T}_{w^{-1}})$ ($f \in \mathbb{C}_q[G]$).

PROOF. The statement (i) follows from (3.6) and Corollary 2.7. The statement (ii) is a special case of (i). Since $V^*(\lambda)_{w\lambda}$ is one-dimensional, we have $v_{w\lambda}^* \in \mathbb{F}^\times v_\lambda^* \dot{T}_{w^{-1}}$. Hence (iii) also follows from Corollary 2.7. \square

COROLLARY 3.20. *The linear map $\mathbb{C}_q[N^+ \setminus G] \ni \varphi \mapsto \varphi \dot{T}_w^{-1} \in \mathbb{C}_q[G]$ is an algebra homomorphism. Hence $\mathbb{C}_q[N^+ \setminus G] \dot{T}_w^{-1}$ is a subalgebra of $\mathbb{C}_q[G]$.*

PROPOSITION 3.21. *Let $f \in \mathbb{C}_q[G]$ and $\lambda \in P^-$.*

- (i) *If $f\sigma_\lambda^w = 0$, then $f = 0$.*
- (ii) *If $\sigma_\lambda^w f = 0$, then $f = 0$.*

PROOF. By Lemma 3.19 we have

$$\begin{aligned} f\sigma_\lambda^w &= (f\dot{T}_w\dot{T}_w^{-1})\sigma_\lambda^w = ((f\dot{T}_w)\sigma_\lambda^1)\dot{T}_w^{-1}, \\ \sigma_\lambda^w f &= \sigma_\lambda^w(f\dot{T}_{w^{-1}}^{-1}\dot{T}_{w^{-1}}) \in \mathbb{F}^\times(\sigma_\lambda^1(f\dot{T}_{w^{-1}}^{-1}))\dot{T}_{w^{-1}}. \end{aligned}$$

Hence the assertion follows from Proposition 3.11. \square

By Proposition 3.11 and Corollary 3.20 we have the following.

PROPOSITION 3.22. *For any $\varphi \in \mathbb{C}_q[N^+ \setminus G] \dot{T}_w^{-1}$ and $\lambda \in P^-$ there exists some $\mu \in P^-$ such that $\sigma_\mu^w \varphi \in (\mathbb{C}_q[N^+ \setminus G] \dot{T}_w^{-1})\sigma_\lambda^w$ and $\varphi\sigma_\mu^w \in \sigma_\lambda^w(\mathbb{C}_q[N^+ \setminus G] \dot{T}_w^{-1})$.*

The following result is proved similarly to [13, Proposition 3.4].

PROPOSITION 3.23. *For any $\psi \in \mathbb{C}_q[G/N^-]$ and $\lambda \in P^-$ there exists some $\mu \in P^-$ such that $\sigma_\mu^w \psi \in \mathbb{C}_q[G/N^-]\sigma_\lambda^w$ and $\psi\sigma_\mu^w \in \sigma_\lambda^w\mathbb{C}_q[G/N^-]$.*

LEMMA 3.24. *For any $f \in \mathbb{C}_q[G]$ there exists some $\lambda \in P^-$ such that $f\sigma_\lambda^w \in \mathbb{C}_q[G/N^-](\mathbb{C}_q[N^+ \setminus G] \dot{T}_w^{-1})$.*

PROOF. By Proposition 3.15 there exists some $\lambda \in P^-$ such that $(f\dot{T}_w)\sigma_\lambda^1 \in \mathbb{C}_q[G/N^-]\mathbb{C}_q[N^+ \setminus G]$. Hence by Lemma 3.19 we have

$$\begin{aligned} f\sigma_\lambda^w &= ((f\dot{T}_w)\sigma_\lambda^1)\dot{T}_w^{-1} \in (\mathbb{C}_q[G/N^-]\mathbb{C}_q[N^+ \setminus G])\dot{T}_w^{-1} \\ &= \mathbb{C}_q[G/N^-](\mathbb{C}_q[N^+ \setminus G] \dot{T}_w^{-1}). \end{aligned}$$

\square

PROPOSITION 3.25. *For any $f \in \mathbb{C}_q[G]$ and $\lambda \in P^-$ there exists some $\mu \in P^-$ such that $\sigma_\mu^w f \in \mathbb{C}_q[G]\sigma_\lambda^w$ and $f\sigma_\mu^w \in \sigma_\lambda^w\mathbb{C}_q[G]$.*

PROOF. We can take $\nu \in P^-$ with $f\sigma_\nu^w \in \mathbb{C}_q[G/N^-](\mathbb{C}_q[N^+ \setminus G] \dot{T}_w^{-1})$ by Lemma 3.24. By Proposition 3.22 and Proposition 3.23 we have $f\sigma_{\nu+\mu'}^w = \sigma_\lambda^w\mathbb{C}_q[G]$ when $\mu' \in P^-$ is sufficiently small. Similarly we have $\sigma_{\mu''}^w f\sigma_\nu^w = \mathbb{C}_q[G]\sigma_{\lambda+\nu}^w$ when $\mu'' \in P^-$ is sufficiently small. Then we have $\sigma_{\mu''}^w f = \mathbb{C}_q[G]\sigma_\lambda^w$ by Proposition 3.21. \square

By Proposition 3.21, Proposition 3.22, Proposition 3.23, and Proposition 3.25 we have the following.

COROLLARY 3.26. *The multiplicative set \mathcal{S}_w satisfies the left and right Ore conditions in all of the three rings $\mathbb{C}_q[G/N^-]$, $\mathbb{C}_q[N^+ \setminus G] \dot{T}_w^{-1}$, $\mathbb{C}_q[G]$.*

It follows that we have the localizations

$$(3.15) \quad \mathcal{S}_w^{-1}\mathbb{C}_q[G/N^-] = \mathbb{C}_q[G/N^-]\mathcal{S}_w^{-1},$$

$$(3.16) \quad \mathcal{S}_w^{-1}(\mathbb{C}_q[N^+ \setminus G] \dot{T}_w^{-1}) = (\mathbb{C}_q[N^+ \setminus G] \dot{T}_w^{-1})\mathcal{S}_w^{-1},$$

$$(3.17) \quad \mathcal{S}_w^{-1}\mathbb{C}_q[G] = \mathbb{C}_q[G]\mathcal{S}_w^{-1}.$$

For $\lambda \in P$ define $\sigma_\lambda^w \in \mathcal{S}_w^{-1}\mathbb{C}_q[G]$ by

$$(3.18) \quad \sigma_\lambda^w = (\sigma_{\lambda_2}^w)^{-1} \sigma_{\lambda_1}^w \quad (\lambda_1, \lambda_2 \in P^-, \lambda = \lambda_1 - \lambda_2),$$

and set

$$(3.19) \quad \tilde{\mathcal{S}}_w = \{\sigma_\lambda^w \mid \lambda \in P\}, \quad \mathbb{F}[\tilde{\mathcal{S}}_w] = \bigoplus_{\lambda \in P} \mathbb{F}\sigma_\lambda^w \subset \mathcal{S}_w^{-1}\mathbb{C}_q[G].$$

Note that $\tilde{\mathcal{S}}_w$ is naturally isomorphic to P as a group.

PROPOSITION 3.27. *We can define a bijective linear map*

$$(3.20) \quad F_w : \mathcal{S}_1^{-1}\mathbb{C}_q[G] \rightarrow \mathcal{S}_w^{-1}\mathbb{C}_q[G]$$

by

$$F_w(f(\sigma_\lambda^1)^{-1}) = (f\dot{T}_w^{-1})(\sigma_\lambda^w)^{-1} \quad (\lambda \in P^-, f \in \mathbb{C}_q[G]).$$

PROOF. Assume $f(\sigma_\lambda^1)^{-1} = f'(\sigma_\mu^1)^{-1}$ ($\lambda, \mu \in P^-, f, f' \in \mathbb{C}_q[G]$). Then we have $f\sigma_\mu^1 = f'\sigma_\lambda^1$, and hence we have $(f\dot{T}_w^{-1})\sigma_\mu^w = (f'\dot{T}_w^{-1})\sigma_\lambda^w$ by Lemma 3.19(ii). It follows that $(f\dot{T}_w^{-1})(\sigma_\lambda^w)^{-1} = (f'\dot{T}_w^{-1})(\sigma_\mu^w)^{-1}$. The bijectivity is obvious. \square

LEMMA 3.28. (i) *We have*

$$\begin{aligned} F_w(\mathcal{S}_1^{-1}\mathbb{C}_q[G/N^-]) &= \mathcal{S}_w^{-1}\mathbb{C}_q[G/N^-], \\ F_w(\mathcal{S}_1^{-1}\mathbb{C}_q[N^+ \setminus G]) &= \mathcal{S}_w^{-1}(\mathbb{C}_q[N^+ \setminus G]\dot{T}_w^{-1}). \end{aligned}$$

(ii) *The linear map $\mathcal{S}_1^{-1}\mathbb{C}_q[N^+ \setminus G] \ni f \mapsto F_w(f) \in \mathcal{S}_w^{-1}(\mathbb{C}_q[N^+ \setminus G]\dot{T}_w^{-1})$ is an algebra isomorphism.*

(iii) *For $\varphi \in \mathcal{S}_1^{-1}\mathbb{C}_q[G/N^-]$ and $\psi \in \mathcal{S}_1^{-1}\mathbb{C}_q[N^+ \setminus G]$ we have $F_w(\varphi\psi) = F_w(\varphi)F_w(\psi)$.*

PROOF. The statements (i) and (ii) are obvious. The statement (iii) is a consequence of Lemma 3.19. \square

By the above arguments we obtain the following results.

PROPOSITION 3.29. *The multiplication induces*

$$\mathcal{S}_w^{-1}\mathbb{C}_q[G/N^-] \otimes_{\mathbb{F}[\tilde{\mathcal{S}}_w]} \mathcal{S}_w^{-1}(\mathbb{C}_q[N^+ \setminus G]\dot{T}_w^{-1}) \cong \mathcal{S}_w^{-1}\mathbb{C}_q[G].$$

PROPOSITION 3.30. *For any $f \in \mathbb{C}_q[G]$ there exists some $\lambda \in P^-$ such that*

$$\sigma_\lambda^w f, f\sigma_\lambda^w \in \mathbb{C}_q[G/N^-](\mathbb{C}_q[N^+ \setminus G]\dot{T}_w^{-1}).$$

3.5. Set

$$(3.21) \quad \mathbb{C}_q[N_w^- \setminus G] = \{\varphi \in \mathbb{C}_q[G] \mid \varphi y = \varepsilon(y)\varphi \ (y \in U^-[\dot{T}_w])\}.$$

Note that

$$\sigma_\lambda^w \in \mathbb{C}_q[G/N^-] \cap \mathbb{C}_q[N_w^- \setminus G] \quad (\lambda \in P^+).$$

PROPOSITION 3.31. *The subspace $\mathbb{C}_q[N_w^- \setminus G]$ of $\mathbb{C}_q[G]$ is a subalgebra of $\mathbb{C}_q[G]$.*

PROOF. Let $\varphi, \psi \in \mathbb{C}_q[N_w^- \setminus G]$. For $y \in U^-[\dot{T}_w] \cap U_{-\gamma}^-$ with $\gamma \in Q^+$ we have

$$(\varphi\psi)y = \sum_{(y)} (\varphi y_{(0)})(\psi y_{(1)}) = (\varphi y)(\psi k_{-\gamma}) = \varepsilon(y)\varphi\psi.$$

by Lemma 2.8. Hence $\varphi\psi \in \mathbb{C}_q[N_w^- \setminus G]$. \square

LEMMA 3.32. *Let $\gamma \in Q^+$ and $y \in U^-[\dot{T}_w] \cap U_{-\gamma}^-$. Then for $\varphi \in \mathbb{C}_q[G]$, $\lambda \in P^-$ we have*

$$(\varphi \sigma_\lambda^w)y = q^{-(w\lambda, \gamma)}(\varphi y)\sigma_\lambda^w.$$

PROOF. By Lemma 2.8 we have

$$(\varphi \sigma_\lambda^w)y = (\varphi y)(\sigma_\lambda^w k_{-\gamma}) = q^{-(w\lambda, \gamma)}(\varphi y)\sigma_\lambda^w.$$

□

By Lemma 3.32 we have the following.

LEMMA 3.33. *For $\varphi \in \mathbb{C}_q[G]$, $\lambda \in P^-$ we have $\varphi \in \mathbb{C}_q[N_w^- \setminus G]$ if and only if $\varphi \sigma_\lambda^w \in \mathbb{C}_q[N_w^- \setminus G]$.*

PROPOSITION 3.34. *The multiplicative set \mathcal{S}_w satisfies the left Ore condition in $\mathbb{C}_q[N_w^- \setminus G]$.*

PROOF. Let $f \in \mathbb{C}_q[N_w^- \setminus G]$, $\lambda \in P^-$. Then we can take $f' \in \mathbb{C}_q[G]$ and $\mu \in P^-$ satisfying $\sigma_\mu^w f = f' \sigma_\lambda^w$. Then by Lemma 3.33 we obtain $f' \in \mathbb{C}_q[N_w^- \setminus G]$. □

We will show later that \mathcal{S}_w also satisfies the right Ore condition in $\mathbb{C}_q[N_w^- \setminus G]$ (see Proposition 3.43 below).

By Proposition 3.34 we have the left localizations

$$\mathcal{S}_w^{-1}\mathbb{C}_q[N_w^- \setminus G], \quad \mathcal{S}_w^{-1}(\mathbb{C}_q[G/N^-] \cap \mathbb{C}_q[N_w^- \setminus G]).$$

PROPOSITION 3.35. *The multiplication of $\mathcal{S}_w^{-1}\mathbb{C}_q[N_w^- \setminus G]$ induces the isomorphism*

$$\begin{aligned} & \mathcal{S}_w^{-1}\mathbb{C}_q[N_w^- \setminus G] \\ & \cong \mathcal{S}_w^{-1}(\mathbb{C}_q[G/N^-] \cap \mathbb{C}_q[N_w^- \setminus G]) \otimes_{\mathbb{F}[\tilde{\mathcal{S}}_w]} \mathcal{S}_w^{-1}(\mathbb{C}_q[N^+ \setminus G]\dot{T}_w^{-1}). \end{aligned}$$

PROOF. We see easily that $\mathbb{C}_q[N^+ \setminus G]\dot{T}_w^{-1} \subset \mathbb{C}_q[N_w^- \setminus G]$. Let us show that $\mathcal{S}_w^{-1}(\mathbb{C}_q[N^+ \setminus G]\dot{T}_w^{-1})$ is a free left $\mathbb{F}[\tilde{\mathcal{S}}_w]$ -module. In the case $w = 1$ this is a consequence of Proposition 3.13. For general w this follows from the case $w = 1$ and Lemma 3.28. Take a basis $\{\psi_j\}_{j \in J}$ of the left free $\mathbb{F}[\tilde{\mathcal{S}}_w]$ -module $\mathcal{S}_w^{-1}(\mathbb{C}_q[N^+ \setminus G]\dot{T}_w^{-1})$. We may assume that $\psi_j \in \mathbb{C}_q[N^+ \setminus G]\dot{T}_w^{-1}$.

Let $f \in \mathcal{S}_w^{-1}\mathbb{C}_q[G]$. By Proposition 3.29 we can uniquely write

$$f = \sum_{j \in J_0} \varphi_j \psi_j \quad (\varphi_j \in \mathcal{S}_w^{-1}\mathbb{C}_q[G/N^-]),$$

where J_0 is a finite subset of J . Then we need to show

$$f \in \mathcal{S}_w^{-1}\mathbb{C}_q[N_w^- \setminus G] \iff \varphi_j \in \mathcal{S}_w^{-1}(\mathbb{C}_q[G/N^-] \cap \mathbb{C}_q[N_w^- \setminus G]) \quad (\forall j \in J_0).$$

Assume that $\varphi_j \in \mathcal{S}_w^{-1}(\mathbb{C}_q[G/N^-] \cap \mathbb{C}_q[N_w^- \setminus G])$ for any $j \in J_0$. We can take $\lambda \in P^-$ such that $\sigma_\lambda^w \varphi_j \in \mathbb{C}_q[G/N^-] \cap \mathbb{C}_q[N_w^- \setminus G]$ for any $j \in J_0$. Then from

$$\sigma_\lambda^w f = \sum_{j \in J_0} (\sigma_\lambda^w \varphi_j) \psi_j \in \mathbb{C}_q[N_w^- \setminus G]$$

we obtain $f \in \mathcal{S}_w^{-1}\mathbb{C}_q[N_w^- \setminus G]$.

Assume that $f \in \mathcal{S}_w^{-1}\mathbb{C}_q[N_w^- \setminus G]$. Taking $\lambda \in P^-$ such that $\sigma_\lambda^w \varphi_j \in \mathbb{C}_q[G/N^-]$ for any $j \in J_0$, we have

$$\sigma_\lambda^w f = \sum_{j \in J_0} (\sigma_\lambda^w \varphi_j) \psi_j \quad (\sigma_\lambda^w \varphi_j \in \mathbb{C}_q[G/N^-]).$$

By $f \in \mathcal{S}_w^{-1}\mathbb{C}_q[N_w^- \setminus G]$ we may assume that $\sigma_\lambda^w f \in \mathbb{C}_q[N_w^- \setminus G]$. Then by Lemma 2.8 we have

$$\varepsilon(y) \sigma_\lambda^w f = (\sigma_\lambda^w f) y = \sum_{j \in J_0} ((\sigma_\lambda^w \varphi_j) y) \psi_j \quad (y \in U^-[\dot{T}_w]).$$

By $(\sigma_\lambda^w \varphi_j) y \in \mathbb{C}_q[G/N^-]$ we have $(\sigma_\lambda^w \varphi_j) y = \varepsilon(y)(\sigma_\lambda^w \varphi_j)$ for any $j \in J_0$, and hence $\sigma_\lambda^w \varphi_j \in \mathbb{C}_q[G/N^-] \cap \mathbb{C}_q[N_w^- \setminus G]$. It follows that $\varphi_j \in \mathcal{S}_w^{-1}(\mathbb{C}_q[G/N^-] \cap \mathbb{C}_q[N_w^- \setminus G])$ for any $j \in J_0$. \square

3.6. By Proposition 3.13 we have

$$\mathcal{S}_1^{-1}\mathbb{C}_q[G/N^-] \cong (U^+)^\star \otimes \mathbb{C}_q[H].$$

Hence the linear isomorphism $F_w : \mathcal{S}_1^{-1}\mathbb{C}_q[G/N^-] \rightarrow \mathcal{S}_w^{-1}\mathbb{C}_q[G/N^-]$ induces an isomorphism

$$(3.22) \quad \mathcal{S}_w^{-1}\mathbb{C}_q[G/N^-] \cong F_w((U^+)^\star) \otimes_{\mathbb{F}} \mathbb{F}[\tilde{\mathcal{S}}_w] \quad (f \sigma_\lambda^w \leftrightarrow f \otimes \sigma_\lambda^w)$$

of vector spaces.

In this subsection we are going to show the following.

PROPOSITION 3.36. *We have*

$$\begin{aligned} & \mathcal{S}_w^{-1}(\mathbb{C}_q[G/N^-] \cap \mathbb{C}_q[N_w^- \setminus G]) \\ &= \left\{ F_w((U^+)^\star) \cap \mathcal{S}_w^{-1}\mathbb{C}_q[N_w^- \setminus G] \right\} \otimes_{\mathbb{F}} \mathbb{F}[\tilde{\mathcal{S}}_w]. \end{aligned}$$

Let $\varphi \in (U^+)^\star$. Then for any sufficiently small $\lambda \in P^-$ there a unique $v^* \in V^*(\lambda)$ such that

$$\langle v^*, x v_\lambda \rangle = \langle \varphi, x \rangle \quad (x \in U^+)$$

by Proposition 2.2. We denote this v^* by $v^*(\varphi, \lambda)$.

LEMMA 3.37. *Let $\varphi \in (U^+)^\star$. Then for sufficiently small $\lambda \in P^-$ we have*

$$F_w(\varphi) = \Phi_{v^*(\varphi, \lambda) \dot{T}_w^{-1} \otimes v_\lambda}(\sigma_\lambda^w)^{-1}.$$

PROOF. For $x \in U^+, t \in U^0, y \in U^-$ we have

$$\begin{aligned} \langle \Phi_{v^*(\varphi, \lambda) \otimes v_\lambda}, xty \rangle &= \langle v^*(\varphi, \lambda), xty v_\lambda \rangle = \langle v^*(\varphi, \lambda), x v_\lambda \rangle \chi_\lambda(t) \varepsilon(y) \\ &= \langle \varphi, x \rangle \chi_\lambda(t) \varepsilon(y). \end{aligned}$$

Hence we obtain $\Phi_{v^*(\varphi, \lambda) \otimes v_\lambda} = \varphi \chi_\lambda$, or equivalently, $\varphi = \Phi_{v^*(\varphi, \lambda) \otimes v_\lambda}(\chi_\lambda)^{-1}$. It follows that

$$F_w(\varphi) = \{ \Phi_{v^*(\varphi, \lambda) \otimes v_\lambda} \dot{T}_w^{-1} \} (\sigma_\lambda^w)^{-1} = \Phi_{v^*(\varphi, \lambda) \dot{T}_w^{-1} \otimes v_\lambda}(\sigma_\lambda^w)^{-1}.$$

\square

COROLLARY 3.38. $F_w((U^+)^\star) = \bigcup_{\lambda \in P^-} \{ \Phi_{v^* \otimes v_\lambda}(\sigma_\lambda^w)^{-1} \mid v^* \in V^*(\lambda) \}.$

LEMMA 3.39. *For $\mu \in P$ we have $\sigma_\mu^w F_w((U^+)^\star) = F_w((U^+)^\star) \sigma_\mu^w$.*

PROOF. We may assume that $\mu \in P^-$. For $\lambda \in P^-$, $v^* \in V^*(\lambda)$ we have

$$\begin{aligned}\sigma_\mu^w \Phi_{v^* \otimes v_\lambda} &= \Phi_{v_{w\mu}^* \otimes v_\mu} \Phi_{v^* \otimes v_\lambda} = \Phi_{(v_{w\mu}^* \otimes v^*) \otimes (v_\mu \otimes v_\lambda)}, \\ \Phi_{v^* \otimes v_\lambda} \sigma_\mu^w &= \Phi_{v^* \otimes v_\lambda} \Phi_{v_{w\mu}^* \otimes v_\mu} = \Phi_{(v^* \otimes v_{w\mu}^*) \otimes (v_\lambda \otimes v_\mu)}.\end{aligned}$$

Let

$$p : V^*(\mu) \otimes V^*(\lambda) \rightarrow V^*(\lambda + \mu), \quad p' : V^*(\lambda) \otimes V^*(\mu) \rightarrow V^*(\lambda + \mu)$$

be the homomorphisms of right U -modules such that $p(v_\lambda^* \otimes v_\mu^*) = v_{\lambda+\mu}^*$, $p'(v_\mu^* \otimes v_\lambda^*) = v_{\lambda+\mu}^*$. Then by [13, Lemma 3.5] we have

$$p(v_{w\mu}^* \otimes V^*(\lambda)_{\lambda+\gamma}) = V^*(\lambda + \mu)_{w\mu+\lambda+\gamma} = p'(V^*(\lambda)_{\lambda+\gamma} \otimes v_{w\mu}^*)$$

for $\gamma \in Q^+$ if $\lambda \in P^-$ is sufficiently small. Hence the assertion follows from Corollary 3.38. \square

LEMMA 3.40. Let $\gamma, \delta \in Q^+$, and let $\varphi \in (U_\delta^+)^*$, $y \in U^-[\dot{T}_w] \cap U_{-\gamma}^-$. Take $z \in U_{-w^{-1}\gamma}^+$ such that $y = \dot{T}_w(k_{-w^{-1}\gamma}z)$ (see (2.34)), and define $\varphi^z \in (U^+)^\star$ by

$$\langle \varphi^z, x \rangle = \langle \varphi, zx \rangle \quad (x \in U^+).$$

If $\lambda \in P^-$ is sufficiently small, then we have $F_w(\varphi)\sigma_\lambda^w \in \mathbb{C}_q[G/N^-]$, and

$$(F_w(\varphi)\sigma_\lambda^w)y = q^{-(w^{-1}\gamma, \lambda+\delta)}F_w(\varphi^z)\sigma_\lambda^w.$$

PROOF. If $\lambda \in P^-$ is sufficiently small, then we have $F_w(\varphi)\sigma_\lambda^w = \Phi_{v^*(\varphi, \lambda)\dot{T}_w^{-1} \otimes v_\lambda} \in \mathbb{C}_q[G/N^-]$. For $x \in U^+$ we have

$$\langle v^*(\varphi, \lambda)z, xv_\lambda \rangle = \langle v^*(\varphi, \lambda), zxv_\lambda \rangle = \langle \varphi, zx \rangle = \langle \varphi^z, x \rangle$$

and hence $v^*(\varphi, \lambda)z = v^*(\varphi^z, \lambda)$. It follows that

$$\begin{aligned}(F_w(\varphi)\sigma_\lambda^w)y &= \Phi_{v^*(\varphi, \lambda)\dot{T}_w^{-1} \otimes v_\lambda} y = \Phi_{v^*(\varphi, \lambda)k_{-w^{-1}\gamma}z\dot{T}_w^{-1} \otimes v_\lambda} \\ &= q^{-(w^{-1}\gamma, \lambda+\delta)}\Phi_{v^*(\varphi^z, \lambda)\dot{T}_w^{-1} \otimes v_\lambda} = q^{-(w^{-1}\gamma, \lambda+\delta)}F_w(\varphi^z)\sigma_\lambda^w.\end{aligned}$$

\square

Let us give a proof of Proposition 3.36. By (3.22) any $f \in \mathcal{S}_w^{-1}\mathbb{C}_q[G/N^-]$ is uniquely written as

$$f = \sum_{\lambda \in P} F_w(\varphi_\lambda)\sigma_\lambda^w \in \mathcal{S}_w^{-1}\mathbb{C}_q[G/N^-] \quad (\varphi_\lambda \in (U^+)^\star).$$

We need to show that $f \in \mathcal{S}_w^{-1}(\mathbb{C}_q[G/N^-] \cap \mathbb{C}_q[N_w^- \setminus G])$ if and only if $F_w(\varphi_\lambda) \in \mathcal{S}_w^{-1}\mathbb{C}_q[N_w^- \setminus G]$ for any $\lambda \in P$. By Lemma 3.33 we have

$$\begin{aligned}f &\in \mathcal{S}_w^{-1}(\mathbb{C}_q[G/N^-] \cap \mathbb{C}_q[N_w^- \setminus G]) \\ \iff \exists \nu \in P^- \text{ s.t. } \sigma_\nu^w f &\in \mathbb{C}_q[G/N^-] \cap \mathbb{C}_q[N_w^- \setminus G] \\ \iff \exists \nu \in P^- \text{ s.t. } \sigma_\nu^w f &\in \mathbb{C}_q[G/N^-], \sigma_\nu^w f \sigma_\mu^w \in \mathbb{C}_q[G/N^-] \cap \mathbb{C}_q[N_w^- \setminus G] \\ \iff f \sigma_\mu^w &\in \mathcal{S}_w^{-1}(\mathbb{C}_q[G/N^-] \cap \mathbb{C}_q[N_w^- \setminus G])\end{aligned}$$

for any $\mu \in P^-$. Hence we may assume from the beginning that f is written as

$$f = \sum_{\lambda \in P^-} F_w(\varphi_\lambda)\sigma_\lambda^w \quad (\varphi_\lambda \in (U^+)^\star).$$

If $F_w(\varphi_\lambda) \in \mathcal{S}_w^{-1}\mathbb{C}_q[N_w^-\backslash G]$ for any $\lambda \in P^-$, there exists some $\mu \in P^-$ such that $\sigma_\mu^w F_w(\varphi_\lambda) \in \mathbb{C}_q[G/N^-] \cap \mathbb{C}_q[N_w^-\backslash G]$ for any $\lambda \in P^-$. It follows that

$$\sigma_\mu^w f = \sum_{\lambda \in P^-} (\sigma_\mu^w F_w(\varphi_\lambda)) \sigma_\lambda^w \in \mathbb{C}_q[G/N^-] \cap \mathbb{C}_q[N_w^-\backslash G]$$

by Lemma 3.33, and hence $f \in \mathcal{S}_w^{-1}(\mathbb{C}_q[G/N^-] \cap \mathbb{C}_q[N_w^-\backslash G])$.

It remains to show that if $f \in \mathcal{S}_w^{-1}(\mathbb{C}_q[G/N^-] \cap \mathbb{C}_q[N_w^-\backslash G])$, then $F_w(\varphi_\lambda) \in \mathcal{S}_w^{-1}\mathbb{C}_q[N_w^-\backslash G]$ for any $\lambda \in P^-$. So assume that $f \in \mathcal{S}_w^{-1}(\mathbb{C}_q[G/N^-] \cap \mathbb{C}_q[N_w^-\backslash G])$. Take $\mu \in P^-$ which is sufficiently small. Then we have $\sigma_\mu^w f \in \mathbb{C}_q[G/N^-] \cap \mathbb{C}_q[N_w^-\backslash G]$. By Lemma 3.39 we can write

$$\sigma_\mu^w F_w(\varphi_\lambda) = F_w(\varphi'_\lambda) \sigma_\mu^w \in \mathbb{C}_q[G/N^-] \quad (\lambda \in P^-, \varphi'_\lambda \in (U^+)^\star),$$

and hence

$$\sigma_\mu^w f = \sum_{\lambda \in P^-} F_w(\varphi'_\lambda) \sigma_{\mu+\lambda}^w, \quad F_w(\varphi'_\lambda) \sigma_{\mu+\lambda}^w \in \mathbb{C}_q[G/N^-] \quad (\lambda \in P^-).$$

Let $\gamma \in Q^+ \setminus \{0\}$ and $y \in U^-[\dot{T}_w] \cap U_{-\gamma}^-$. By $\sigma_\mu^w f \in \mathbb{C}_q[N_w^-\backslash G]$ we have $(\sigma_\mu^w f)y = 0$. On the other hand we have

$$(\sigma_\mu^w f)y = \sum_{\lambda \in P^-} (F_w(\varphi'_\lambda) \sigma_{\lambda+\mu}^w)y = \sum_{\lambda \in P^-} q^{-(w\lambda, \gamma)} ((F_w(\varphi'_\lambda) \sigma_\mu^w)y) \sigma_\lambda^w.$$

By Lemma 3.40 we have

$$(F_w(\varphi'_\lambda) \sigma_\mu^w)y = F_w(\varphi''_\lambda) \sigma_\mu^w$$

for some $\varphi''_\lambda \in (U^+)^\star$, and hence

$$\sum_{\lambda \in P^-} q^{-(w\lambda, \gamma)} F_w(\varphi''_\lambda) \sigma_{\lambda+\mu}^w = 0$$

By (3.22) we obtain $F_w(\varphi''_\lambda) = 0$ for any $\lambda \in P^-$. It follows that

$$(\sigma_\mu^w F_w(\varphi_\lambda))y = (F_w(\varphi'_\lambda) \sigma_\mu^w)y = F_w(\varphi''_\lambda) \sigma_\mu^w = 0.$$

We obtain $F_w(\varphi_\lambda) \in \mathcal{S}_w^{-1}\mathbb{C}_q[N_w^-\backslash G]$ for any $\lambda \in P^-$. The proof of Proposition 3.36 is complete.

3.7. Set

$$\begin{aligned} \mathcal{J}_w &= \{\psi \in (U^+)^\star \mid \psi^z = \varepsilon(z)\psi \quad (z \in U^+[\dot{T}_w^{-1}])\} \\ &= \{\psi \in (U^+)^\star \mid \psi|_{\text{Ker}(\varepsilon: U^+[\dot{T}_w^{-1}] \rightarrow \mathbb{F})U^+} = 0\} \end{aligned}$$

In this subsection we are going to show the following.

PROPOSITION 3.41. $F_w((U^+)^\star) \cap \mathcal{S}_w^{-1}\mathbb{C}_q[N_w^-\backslash G] = F_w(\mathcal{J}_w)$.

We first show the following result.

LEMMA 3.42. *Let $\gamma \in Q^+$, $\psi \in \mathcal{J}_w \cap (U_\gamma^+)^\star$ and $\mu \in P$. Then we have $q^{(\mu, \gamma)} F_w(\psi) \sigma_\mu^w = \sigma_\mu^w F_w(\psi)$.*

PROOF. We may assume $\mu \in P^-$. When $\lambda \in P^-$ is sufficiently small, we have $F_w(\psi) = \Phi_{v^*(\psi, \lambda) \dot{T}_w^{-1} \otimes v_\lambda}(\sigma_\lambda^w)^{-1}$, and hence it is sufficient to show

$$q^{(\mu, \gamma)} \Phi_{v^*(\psi, \lambda) \dot{T}_w^{-1} \otimes v_\lambda} \sigma_\mu^w = \sigma_\mu^w \Phi_{v^*(\psi, \lambda) \dot{T}_w^{-1} \otimes v_\lambda}.$$

We have

$$\begin{aligned} \Phi_{v^*(\psi, \lambda) \dot{T}_w^{-1} \otimes v_\lambda} \sigma_\mu^w &= \Phi_{v^*(\psi, \lambda) \dot{T}_w^{-1} \otimes v_\lambda} \Phi_{v_\mu^* \dot{T}_w^{-1} \otimes v_\mu} = \Phi_{(v^*(\psi, \lambda) \dot{T}_w^{-1} \otimes v_\mu^* \dot{T}_w^{-1}) \otimes (v_\lambda \otimes v_\mu)}, \\ \sigma_\mu^w \Phi_{v^*(\psi, \lambda) \dot{T}_w^{-1} \otimes v_\lambda} &= \Phi_{v_\mu^* \dot{T}_w^{-1} \otimes v_\mu} \Phi_{v^*(\psi, \lambda) \dot{T}_w^{-1} \otimes v_\lambda} = \Phi_{(v_\mu^* \dot{T}_w^{-1} \otimes v^*(\psi, \lambda) \dot{T}_w^{-1}) \otimes (v_\mu \otimes v_\lambda)}. \end{aligned}$$

Since v_μ^* is the lowest weight vector we have

$$v^*(\psi, \lambda) \dot{T}_w^{-1} \otimes v_\mu^* \dot{T}_w^{-1} = (v^*(\psi, \lambda) \otimes v_\mu^*) \dot{T}_w^{-1}.$$

On the other hand by $\psi \in \mathcal{J}_w$ we have

$$v^*(\psi, \lambda)z = \varepsilon(z)v^*(\psi, \lambda) \quad (z \in U^+[\dot{T}_w^{-1}]),$$

and hence

$$v_\mu^* \dot{T}_w^{-1} \otimes v^*(\psi, \lambda) \dot{T}_w^{-1} = (v_\mu^* \otimes v^*(\psi, \lambda)) \dot{T}_w^{-1}.$$

Therefore, we have only to show

$$q^{(\mu, \gamma)} \Phi_{(v^*(\psi, \lambda) \otimes v_\mu^*) \dot{T}_w^{-1} \otimes (v_\lambda \otimes v_\mu)} = \Phi_{(v_\mu^* \otimes v^*(\psi, \lambda)) \dot{T}_w^{-1} \otimes (v_\mu \otimes v_\lambda)}.$$

Let

$$p : V^*(\lambda) \otimes V^*(\mu) \rightarrow V^*(\lambda + \mu), \quad p' : V^*(\mu) \otimes V^*(\lambda) \rightarrow V^*(\lambda + \mu)$$

be the homomorphisms of U -modules such that $p(v_\lambda^* \otimes v_\mu^*) = v_{\lambda+\mu}^*$ and $p'(v_\mu^* \otimes v_\lambda^*) = v_{\lambda+\mu}^*$. The our assertion is equivalent to

$$q^{(\mu, \gamma)} p(v^*(\psi, \lambda) \otimes v_\mu^*) = q^{(\mu, \gamma)} v^*(\psi, \lambda + \mu) = p'(v_\mu^* \otimes v^*(\psi, \lambda)).$$

This follows from

$$\begin{aligned} \langle (v^*(\psi, \lambda) \otimes v_\mu^*)x, v_\lambda \otimes v_\mu \rangle &= \langle v^*(\psi, \lambda)x \otimes v_\mu^*, v_\lambda \otimes v_\mu \rangle = \langle \psi, x \rangle, \\ \langle (v_\mu^* \otimes v^*(\psi, \lambda))x, v_\mu \otimes v_\lambda \rangle &= \langle v_\mu^* k_\gamma \otimes v^*(\psi, \lambda)x, v_\mu \otimes v_\lambda \rangle = q^{(\mu, \gamma)} \langle \psi, x \rangle \end{aligned}$$

for $x \in U^+$. □

Let us give a proof of Proposition 3.41. Assume that $\varphi \in (U^+)^\star$ satisfies $F_w(\varphi) \in \mathcal{S}_w^{-1} \mathbb{C}_q[N_w^- \setminus G]$. When $\mu \in P^-$ is sufficiently small, we have $\sigma_\mu^w F_w(\varphi) \in \mathbb{C}_q[N_w^- \setminus G]$. By Lemma 3.39 we have

$$(3.23) \quad \sigma_\mu^w F_w(\varphi) = F_w(\varphi') \sigma_\mu^w.$$

By Lemma 3.40 we have

$$(\varphi')^z = \varepsilon(z) \varphi' \quad (z \in U^+[\dot{T}_w^{-1}]),$$

namely $\varphi' \in \mathcal{J}_w$. Hence (3.23) and Lemma 3.42 implies $\varphi \in \mathcal{J}_w$.

Assume conversely that $\varphi \in \mathcal{J}_w$. Then by Lemma 3.40 and Lemma 3.42 we have $F_w(\varphi) \in \mathcal{S}_w^{-1} \mathbb{C}_q[N_w^- \setminus G]$.

The proof of Proposition 3.41 is complete.

3.8. By Proposition 3.35, Corollary 3.26, Proposition 3.36, Proposition 3.41, and Lemma 3.42 we obtain the following.

PROPOSITION 3.43. *The multiplicative set \mathcal{S}_w satisfies the right Ore condition in $\mathbb{C}_q[N_w^- \setminus G]$.*

Set

$$(3.24) \quad U^+[\dot{T}_w^{-1}]^\star = \sum_{\gamma \in Q^+} (U^+[\dot{T}_w^{-1}] \cap U_\gamma^+)^* \subset (U^+[\dot{T}_w^{-1}])^*.$$

In view of (2.16) we can define an injective linear map

$$(3.25) \quad i_w^+ : U^+[\dot{T}_w^{-1}]^\star \rightarrow (U^+)^\star$$

by

$$\langle i_w^+(\varphi), x_1 x_2 \rangle = \langle \varphi, u_1 \rangle \varepsilon(u_2) \quad (x_1 \in U^+[\dot{T}_w^{-1}], x_2 \in U^+ \cap \dot{T}_w^{-1}(U^+)).$$

PROPOSITION 3.44. (i) *The multiplication of $(U^+)^\star$ (as a subalgebra of U^*) induces an isomorphism*

$$i_w^+(U^+[\dot{T}_w^{-1}]^\star) \otimes \mathcal{J}_w \cong (U^+)^\star$$

of vector spaces.

(ii) *For $\varphi \in U^+[\dot{T}_w^{-1}]^\star$, $\psi \in \mathcal{J}_w$ we have*

$$F_w(i_w^+(\varphi)\psi) = F_w(i_w^+(\varphi))F_w(\psi) \quad (\varphi \in U^+[\dot{T}_w^{-1}]^\star, \psi \in \mathcal{J}_w).$$

PROOF. (i) For $\varphi \in U^+[\dot{T}_w^{-1}]^\star$, $\psi \in \mathcal{J}_w$, $x \in U^+[\dot{T}_w^{-1}]$, $x' \in U^+ \cap \dot{T}_w^{-1}U^{\geq 0}$ we have

$$\langle i_w^+(\varphi)\psi, xx' \rangle = \sum_{(x), (x')} \langle i_w^+(\varphi), x_{(0)}x'_{(0)} \rangle \langle \psi, x_{(1)}x'_{(1)} \rangle.$$

Hence by Lemma 2.8 we obtain

$$\langle i_w^+(\varphi)\psi, xx' \rangle = \langle i_w^+(\varphi), x \rangle \langle \psi, x' \rangle.$$

(ii) Take $\lambda \in P^-$ such that $i_w^+(\varphi)\chi_\lambda \in \mathbb{C}_q[G/N^-]$. Then we have $\chi_\lambda^{-1}\psi\chi_\lambda = \psi' \in \mathcal{J}_w$. Take $\mu \in P^-$ such that $\psi'\chi_\mu \in \mathbb{C}_q[G/N^-]$. We may assume that $\psi'\chi_\mu = \Phi_{v^* \otimes v_\nu}$ and

$$v^*z = \varepsilon(z)v^* \quad (z \in U^+[\dot{T}_w^{-1}]).$$

Then we have

$$\begin{aligned} F_w(i_w^+(\varphi)\psi) &= F_w((i_w^+(\varphi)\chi_\lambda)(\psi'\chi_\mu)\chi_{\lambda+\mu}^{-1}) \\ &= \{ \{ (i_w^+(\varphi)\chi_\lambda)(\psi'\chi_\mu) \} \dot{T}_w^{-1} \} (\sigma_{\lambda+\mu}^w)^{-1} \\ &= \{ (i_w^+(\varphi)\chi_\lambda) \dot{T}_w^{-1} \} \{ (\psi'\chi_\mu) \dot{T}_w^{-1} \} (\sigma_{\lambda+\mu}^w)^{-1} \\ &= \{ F_w(i_w^+(\varphi)) \sigma_\lambda^w \} \{ F_w(\psi') \sigma_\mu^w \} (\sigma_{\lambda+\mu}^w)^{-1} = F_w(i_w^+(\varphi))F_w(\psi'). \end{aligned}$$

Here, the last equality is a consequence of Lemma 3.42. \square

4. INDUCED MODULES

4.1. We fix $w \in W$ in this section.

By Proposition 2.12, Corollary 2.7 and (3.6) we have

$$(4.1) \quad (\varphi\psi)\dot{T}_w = (\varphi\dot{T}_w)(\psi\dot{T}_w) \quad (\varphi \in \mathbb{C}_q[G], \psi \in \mathbb{C}_q[N_w^- \setminus G]).$$

Define $\eta'_w : \mathbb{C}_q[N_w^- \setminus G] \rightarrow \mathbb{C}_q[H]$ by

$$\langle \eta'_w(\varphi), t \rangle = \langle \varphi\dot{T}_w, t \rangle (= \langle \dot{T}_w\varphi, \dot{T}_w(t) \rangle) \quad (t \in U^0)$$

(see Lemma 3.16).

LEMMA 4.1. *The linear map η'_w is an algebra homomorphism. Moreover, for $\lambda \in P^-$ we have $\eta'_w(\sigma_\lambda^w) = \chi_\lambda \in \mathbb{C}_q[H]^\times$.*

PROOF. For $\varphi, \psi \in \mathbb{C}_q[N_w^- \setminus G], t \in U^0$ we have

$$\begin{aligned} \langle \eta'_w(\varphi\psi), t \rangle &= \langle (\varphi\psi)\dot{T}_w, t \rangle = \langle (\varphi\dot{T}_w)(\psi\dot{T}_w), t \rangle \\ &= \sum_{(t)} \langle \varphi\dot{T}_w, t_{(0)} \rangle \langle \psi\dot{T}_w, t_{(1)} \rangle = \langle \eta'_w(\varphi)\eta'_w(\psi), t \rangle. \end{aligned}$$

by (4.1). For $\lambda \in P^-$ and $t \in U^0$ we have

$$\langle \eta'_w(\sigma_\lambda^w), t \rangle = \langle \sigma_\lambda^w\dot{T}_w, t \rangle = \langle v_\lambda^*, tv_\lambda \rangle = \langle \chi_\lambda, t \rangle.$$

□

Hence we obtain an algebra homomorphism

$$(4.2) \quad \eta_w : \mathcal{S}_w^{-1}\mathbb{C}_q[N_w^- \setminus G] \rightarrow \mathbb{C}_q[H]$$

by extending η'_w .

DEFINITION 4.2. Define an $(\mathcal{S}_w^{-1}\mathbb{C}_q[G], \mathbb{C}_q[H])$ -bimodule \mathcal{M}_w by

$$(4.3) \quad \mathcal{M}_w = \mathcal{S}_w^{-1}\mathbb{C}_q[G] \otimes_{\mathcal{S}_w^{-1}\mathbb{C}_q[N_w^- \setminus G]} \mathbb{C}_q[H],$$

where $\mathcal{S}_w^{-1}\mathbb{C}_q[N_w^- \setminus G] \rightarrow \mathbb{C}_q[H]$ is given by η_w .

By

$$\mathcal{S}_w^{-1}\mathbb{C}_q[G] = \mathbb{C}_q[G] \otimes_{\mathbb{C}_q[N_w^- \setminus G]} \mathcal{S}_w^{-1}\mathbb{C}_q[N_w^- \setminus G]$$

we have

$$(4.4) \quad \mathcal{M}_w \cong \mathbb{C}_q[G] \otimes_{\mathbb{C}_q[N_w^- \setminus G]} \mathbb{C}_q[H].$$

For $\varphi \in \mathcal{S}_w^{-1}\mathbb{C}_q[G]$ and $\chi \in \mathbb{C}_q[H]$ we write

$$(4.5) \quad \varphi \star \chi := \varphi \otimes \chi \in \mathcal{M}_w.$$

Then we have

$$(4.6) \quad \varphi\sigma_\lambda^w \star \chi = \varphi \star \chi\lambda\chi \quad (\varphi \in \mathcal{S}_w^{-1}\mathbb{C}_q[G], \lambda \in P, \chi \in \mathbb{C}_q[H]).$$

By (4.4) \mathcal{M}_w is generated by $\{\varphi \star 1 \mid \varphi \in \mathbb{C}_q[G]\}$ as a $\mathbb{C}_q[H]$ -module.

4.2. Set

$$U^{\geq 0}[\dot{T}_w^{-1}] = (U^+[\dot{T}_w^{-1}])U^0 \subset U^{\geq 0}.$$

Define an injective linear map

$$U^+[\dot{T}_w^{-1}]^\star \otimes \mathbb{C}_q[H] \rightarrow \text{Hom}_{\mathbb{F}}(U^{\geq 0}[\dot{T}_w^{-1}], \mathbb{F}) \quad (f \otimes \chi \mapsto c_{f \otimes \chi})$$

by

$$\langle c_{f \otimes \chi}, xt \rangle = \langle f, x \rangle \langle \chi, t \rangle \quad (x \in U^+[\dot{T}_w^{-1}], t \in U^0),$$

and denote its image by $U^{\geq 0}[\dot{T}_w^{-1}]^\star$. Then we have an identification

$$(4.7) \quad U^{\geq 0}[\dot{T}_w^{-1}]^\star \cong U^+[\dot{T}_w^{-1}]^\star \otimes \mathbb{C}_q[H] \quad (c_{f \otimes \chi} \leftrightarrow f \otimes \chi)$$

of vector spaces. Since $U^+[\dot{T}_w^{-1}]^\star \otimes \mathbb{C}_q[H]$ is naturally a right $\mathbb{C}_q[H]$ -module by the multiplication of $\mathbb{C}_q[H]$, $U^{\geq 0}[\dot{T}_w^{-1}]^\star$ is also endowed with a right $\mathbb{C}_q[H]$ -module structure via the identification (4.7). Then we have

$$(4.8) \quad \langle f\chi, xt \rangle = \sum_{(t)} \langle f, xt_{(0)} \rangle \langle \chi, t_{(1)} \rangle$$

$$(f \in U^{\geq 0}[\dot{T}_w^{-1}]^\star, \chi \in \mathbb{C}_q[H], x \in U^+[\dot{T}_w^{-1}], t \in U^0).$$

4.3. We construct an isomorphism

$$\Theta_w : \mathcal{M}_w \rightarrow U^{\geq 0}[\dot{T}_w^{-1}]^\star$$

of right $\mathbb{C}_q[H]$ -modules. We first define $\Theta'_w : \mathbb{C}_q[G] \rightarrow U^{\geq 0}[\dot{T}_w^{-1}]^\star$ by

$$\langle \Theta'_w(\varphi), u \rangle = \langle \varphi \dot{T}_w, u \rangle \quad (\varphi \in \mathbb{C}_q[G], u \in U^{\geq 0}[\dot{T}_w^{-1}]).$$

LEMMA 4.3. $\Theta'_w(\varphi\psi) = \Theta'_w(\varphi)\eta'_w(\psi) \quad (\varphi \in \mathbb{C}_q[G], \psi \in \mathbb{C}_q[N_w^- \setminus G]).$

PROOF. Let $\gamma \in Q^+$, $x \in U^+[\dot{T}_w^{-1}] \cap U_\gamma^+$, $t \in U^0$. Then we have

$$\langle \Theta'_w(\varphi), xt \rangle = \langle \varphi \dot{T}_w, xt \rangle = \langle \varphi \dot{T}_w(x) \dot{T}_w, t \rangle.$$

Similarly,

$$\langle \Theta'_w(\varphi\psi), xt \rangle = \langle \{(\varphi\psi) \dot{T}_w(x)\} \dot{T}_w, t \rangle.$$

By (2.35) we can write $\dot{T}_w(x) = yk_{-w\gamma} \quad (y \in U^-[\dot{T}_w] \cap U_{w\gamma}^-)$. Thus by Lemma 2.8 and $\psi \in \mathbb{C}_q[N_w^- \setminus G]$ we have

$$(\varphi\psi)(\dot{T}_w(x)) = \left(\sum_{(y)} (\varphi y_{(0)}) (\psi y_{(1)}) \right) k_{-w\gamma} = \{(\varphi y)(\psi k_{w\gamma})\} k_{-w\gamma}$$

$$= (\varphi(\dot{T}_w(x)))\psi.$$

Hence by (4.1) we have

$$\langle \Theta'_w(\varphi\psi), xt \rangle = \langle \{(\varphi(\dot{T}_w(x)))\psi\} \dot{T}_w, t \rangle = \langle (\varphi(\dot{T}_w(x)) \dot{T}_w)(\psi \dot{T}_w), t \rangle$$

$$= \sum_{(t)} \langle \varphi(\dot{T}_w(x)) \dot{T}_w, t_{(0)} \rangle \langle \psi \dot{T}_w, t_{(1)} \rangle = \sum_{(t)} \langle \Theta'_w(\varphi), xt_{(0)} \rangle \langle \eta'_w(\psi), t_{(1)} \rangle$$

$$= \langle \Theta'_w(\varphi)\eta'_w(\psi), xt \rangle.$$

□

Hence regarding $U^{\geq 0}[\dot{T}_w^{-1}]^\star$ as a right $\mathbb{C}_q[N_w^- \setminus G]$ -module via $\eta'_w : \mathbb{C}_q[N_w^- \setminus G] \rightarrow \mathbb{C}_q[H]$, Θ'_w turns out to be a homomorphism of right $\mathbb{C}_q[N_w^- \setminus G]$ -modules. Moreover, the right action of the elements of $\mathcal{S}_w(\subset \mathbb{C}_q[N_w^- \setminus G])$ on $U^{\geq 0}[\dot{T}_w^{-1}]^\star$ is invertible. Hence Θ'_w induces

$$\Theta''_w : \mathcal{S}_w^{-1}\mathbb{C}_q[G] = \mathbb{C}_q[G] \otimes_{\mathbb{C}_q[N_w^- \setminus G]} \mathcal{S}_w^{-1}\mathbb{C}_q[N_w^- \setminus G] \rightarrow U^{\geq 0}[\dot{T}_w^{-1}]^\star.$$

Then we have

$$(4.9) \quad \Theta''_w(\varphi\psi) = \Theta''_w(\varphi)\eta_w(\psi) \quad (\varphi \in \mathcal{S}_w^{-1}\mathbb{C}_q[G], \psi \in \mathcal{S}_w^{-1}\mathbb{C}_q[N_w^- \setminus G]).$$

Therefore, we obtain a homomorphism

$$(4.10) \quad \Theta_w : \mathcal{M}_w \rightarrow U^{\geq 0}[\dot{T}_w^{-1}]^\star$$

of right $\mathbb{C}_q[H]$ -modules by

$$\Theta_w(\varphi \star \chi) = \Theta''_w(\varphi)\chi \quad (\varphi \in \mathcal{S}_w^{-1}\mathbb{C}_q[G], \chi \in \mathbb{C}_q[H]).$$

PROPOSITION 4.4. *The linear map*

$$\Upsilon_w : U^+[\dot{T}_w^{-1}]^\star \otimes \mathbb{C}_q[H] \rightarrow \mathcal{M}_w \quad (\varphi \otimes \chi \mapsto F_w(i_w^+(\varphi)) \star \chi)$$

is bijective.

PROOF. By Proposition 3.29 and (3.22) we have

$$\mathcal{S}_w^{-1}\mathbb{C}_q[G] \cong F_w((U^+)^\star) \otimes \mathcal{S}_w^{-1}(\mathbb{C}_q[N^+ \setminus G]\dot{T}_w^{-1}).$$

On the other hand by Proposition 3.35, Proposition 3.36, Proposition 3.41 we have

$$\mathcal{S}_w^{-1}\mathbb{C}_q[N_w^- \setminus G] \cong F_w(\mathcal{J}_w) \otimes \mathcal{S}_w^{-1}(\mathbb{C}_q[N^+ \setminus G]\dot{T}_w^{-1}).$$

Hence by Proposition 3.44 we have

$$\mathcal{S}_w^{-1}\mathbb{C}_q[G] \cong F_w(i_w^+(U^+[\dot{T}_w^{-1}]^\star)) \otimes \mathcal{S}_w^{-1}\mathbb{C}_q[N_w^- \setminus G].$$

It follows that $\mathcal{M}_w \cong U^+[\dot{T}_w^{-1}]^\star \otimes \mathbb{C}_q[H]$. □

PROPOSITION 4.5. *We have $\Theta_w \circ \Upsilon_w = \text{id}$ under the identification (4.7). Especially, Θ_w is an isomorphism of right $\mathbb{C}_q[H]$ -modules.*

PROOF. Let $\varphi \in U^+[\dot{T}_w^{-1}]^\star$, $\chi \in \mathbb{C}_q[H]$, $x \in U^+[\dot{T}_w^{-1}]$, $t \in U^0$. Then for $\lambda \in P^-$ which is sufficiently small we have

$$\begin{aligned} & \langle (\Theta_w \circ \Upsilon_w)(\varphi \otimes \chi), xt \rangle = \langle \Theta_w(F_w(i_w^+(\varphi)) \star \chi), xt \rangle \\ &= \langle \Theta_w(\Phi_{v^*(i_w^+(\varphi), \lambda)\dot{T}_w^{-1} \otimes v_\lambda}(\sigma_\lambda^w)^{-1} \star \chi), xt \rangle \\ &= \langle \Theta_w(\Phi_{v^*(i_w^+(\varphi), \lambda)\dot{T}_w^{-1} \otimes v_\lambda} \star \chi - \lambda \chi), xt \rangle \\ &= \langle \Theta'_w(\Phi_{v^*(i_w^+(\varphi), \lambda)\dot{T}_w^{-1} \otimes v_\lambda})(\chi - \lambda \chi), xt \rangle \\ &= \sum_{(t)} \langle \Phi_{v^*(i_w^+(\varphi), \lambda)\dot{T}_w^{-1} \otimes v_\lambda}, xt_{(0)} \rangle \langle \chi - \lambda \chi, t_{(1)} \rangle \\ &= \sum_{(t)_2} \langle i_w^+(\varphi), x \rangle \langle \chi_\lambda, t_{(0)} \rangle \langle \chi - \lambda, t_{(1)} \rangle \langle \chi, t_{(2)} \rangle = \langle \varphi, x \rangle \langle \chi, t \rangle \\ &= \langle \varphi \otimes \chi, x \otimes t \rangle. \end{aligned}$$

□

4.4. In this subsection we consider the special case where $\mathfrak{g} = \mathfrak{sl}_2$ and $G = SL_2$. We follow the notation of Example 3.1. The Weyl group consists of two elements 1 and s . We give below an explicit description of the $(\mathbb{C}_q[G], \mathbb{C}_q[H])$ -bimodule \mathcal{M}_s . For $n \in \mathbb{Z}_{\geq 0}$ define $m(n) \in \mathcal{M}_s$ by

$$\langle \Theta(m(n)), e^{n'} k^i \rangle = \delta_{n,n'} \quad (n' \in \mathbb{Z}_{\geq 0}, i \in \mathbb{Z}).$$

Then we have

$$\mathcal{M}_s = \bigoplus_{n=0}^{\infty} \mathbb{C}_q[H]m(n).$$

LEMMA 4.6. *The action of $\mathbb{C}_q[G]$ on \mathcal{M}_s is given by*

$$\begin{aligned} am(n) &= \chi(q - q^{-1})q^{n-1}m(n-1), & bm(n) &= \chi^{-1}q^n m(n), \\ cm(n) &= -\chi q^{n+1}m(n), & dm(n) &= -\chi^{-1}q[n+1]m(n+1). \end{aligned}$$

PROOF. By Corollary 2.7 we have

$$\begin{aligned} &\langle (\psi\varphi)\dot{T}_s, e^n k^i \rangle \\ &= \sum_{r=0}^n \sum_{p=0}^{\infty} q^{-2ri} c(p, n, r) \langle \psi\dot{T}_s f^{(p)}, k^{n-r+i} e^r \rangle \langle \varphi\dot{T}_s, e^{n+p-r} k^i \rangle \end{aligned}$$

for $\varphi, \psi \in \mathbb{C}_q[G]$, where

$$c(p, n, r) = q^{p(p-1)/2-r(n-r)}(q - q^{-1})^p \begin{bmatrix} n \\ r \end{bmatrix}.$$

By a direct calculation we have

$$a\dot{T}_s = c, \quad a\dot{T}_s f = a, \quad a\dot{T}_s f^{(2)} = 0.$$

Hence for $\varphi \in \mathbb{C}_q[G]$ we have

$$\begin{aligned} &\langle (a\varphi)\dot{T}_s, e^n k^i \rangle \\ &= \sum_{r=0}^n \sum_{p=0}^{\infty} q^{-2ri} c(p, n, r) \langle a\dot{T}_s f^{(p)}, k^{n-r+i} e^r \rangle \langle \varphi\dot{T}_s, e^{n+p-r} k^i \rangle \\ &= \sum_{r=0}^n q^{-2ri} c(0, n, r) \langle c, k^{n-r+i} e^r \rangle \langle \varphi\dot{T}_s, e^{n-r} k^i \rangle \\ &\quad + \sum_{r=0}^n q^{-2ri} c(1, n, r) \langle a, k^{n-r+i} e^r \rangle \langle \varphi\dot{T}_s, e^{n+1-r} k^i \rangle \\ &= c(1, n, 0) q^{n+i} \langle \varphi\dot{T}_s, e^{n+1} k^i \rangle \\ &= q^i (q - q^{-1}) q^n \langle \varphi\dot{T}_s, e^{n+1} k^i \rangle. \end{aligned}$$

Taking $\varphi_j \in \mathbb{C}_q[SL_2]$ such that $m(n) = \sum_j \varphi_j \star \chi^j$ we have

$$\begin{aligned}
\langle \Theta(am(n)), e^{n'} k^i \rangle &= \sum_j \langle \Theta(a\varphi_j \star \chi_j), e^{n'} k^i \rangle \\
&= \sum_j \langle (a\varphi_j) \dot{T}_s, e^{n'} k^i \rangle q^{ij} = q^i (q - q^{-1}) q^{n'} \sum_j \langle \varphi_j \dot{T}_s, e^{n'+1} k^i \rangle q^{ij} \\
&= (q - q^{-1}) q^{n'} \sum_j \langle \Theta(\varphi_j \star \chi^{j+1}), e^{n'+1} k^i \rangle \\
&= (q - q^{-1}) q^{n'} \langle \Theta(m(n)) \chi, e^{n'+1} k^i \rangle = (q - q^{-1}) q^{n'} \delta_{n, n'+1} \langle \chi, k^i \rangle \\
&= (q - q^{-1}) q^{n-1} \langle \Theta(m(n-1)) \chi, e^{n'} k^i \rangle.
\end{aligned}$$

Hence $am(n+1) = \chi(q - q^{-1}) q^n m(n)$. The proof of other formulas are similar. \square

4.5. Let us return to the general situation where \mathfrak{g} is any simple Lie algebra.

PROPOSITION 4.7. *We have*

$$\begin{aligned}
\mathcal{M}_w &\cong \mathcal{S}_w^{-1} \mathbb{C}_q[G/N^-] \otimes_{\mathcal{S}_w^{-1}(\mathbb{C}_q[G/N^-] \cap \mathbb{C}_q[N_w^- \setminus G])} \mathbb{C}_q[H] \\
&\cong \mathbb{C}_q[G/N^-] \otimes_{\mathbb{C}_q[G/N^-] \cap \mathbb{C}_q[N_w^- \setminus G]} \mathbb{C}_q[H].
\end{aligned}$$

PROOF. By (3.22), Proposition 3.36, Proposition 3.41, Proposition 3.44 we have

$$\mathcal{S}_w^{-1} \mathbb{C}_q[G/N^-] \cong F_w(i_w^+(U^+[\dot{T}_w^{-1}]^\star)) \otimes \mathcal{S}_w^{-1}(\mathbb{C}_q[G/N^-] \cap \mathbb{C}_q[N_w^- \setminus G]).$$

Hence we obtain

$$\begin{aligned}
&\mathcal{S}_w^{-1} \mathbb{C}_q[G/N^-] \otimes_{\mathcal{S}_w^{-1}(\mathbb{C}_q[G/N^-] \cap \mathbb{C}_q[N_w^- \setminus G])} \mathbb{C}_q[H] \\
&\cong U^+[\dot{T}_w^{-1}]^\star \otimes \mathbb{C}_q[H] \cong \mathcal{M}_w
\end{aligned}$$

by Proposition 4.4. The second isomorphism is a consequence of

$$\begin{aligned}
&\mathcal{S}_w^{-1} \mathbb{C}_q[G/N^-] \\
&\cong \mathbb{C}_q[G/N^-] \otimes_{\mathbb{C}_q[G/N^-] \cap \mathbb{C}_q[N_w^- \setminus G]} \mathcal{S}_w^{-1}(\mathbb{C}_q[G/N^-] \cap \mathbb{C}_q[N_w^- \setminus G]).
\end{aligned}$$

\square

We regard $\mathbb{C}_q[H]$ as a subalgebra of $\mathbb{C}_q[B^-]$ via the Hopf algebra homomorphism $U^{\leq 0} \rightarrow U^0$ given by $ty \mapsto \varepsilon(y)t$ ($t \in U^0, y \in U^-$). Define an action of W on $\mathbb{C}_q[H]$ by

$$\langle w\chi, t \rangle = \langle \chi, \dot{T}_w^{-1}(t) \rangle \quad (w \in W, \chi \in \mathbb{C}_q[H], t \in U^0).$$

For $w \in W$ we define a twisted right $\mathbb{C}_q[H]$ -module structure of $\mathbb{C}_q[B^-]$ by

$$(4.11) \quad \varphi \bullet_w \chi = (Sw\chi)\varphi \quad (\varphi \in \mathbb{C}_q[B^-], \chi \in \mathbb{C}_q[H]).$$

We denote by $\mathbb{C}_q[B^-]^{\bullet_w}$ the \mathbb{F} -algebra $\mathbb{C}_q[B^-]$ equipped with the twisted right $\mathbb{C}_q[H]$ -module structure (4.11).

We are going to construct an embedding

$$\Xi_w : \mathcal{M}_w \hookrightarrow \mathbb{C}_q[B^-]^{\bullet_w}.$$

of right $\mathbb{C}_q[H]$ -module.

We first define

$$\Xi'_w : \mathbb{C}_q[G/N^-] \rightarrow \mathbb{C}_q[B^-]$$

by

$$\langle \Xi'_w(\varphi), u \rangle = \langle \dot{T}_w \varphi, Su \rangle \quad (\varphi \in \mathbb{C}_q[G/N^-], u \in U^{\leq 0}).$$

LEMMA 4.8. *The linear map Ξ'_w is an algebra anti-homomorphism. Moreover, for $\lambda \in P^-$ we have $\Xi'_w(\sigma_\lambda^w) = \chi_{-w\lambda} \in \mathbb{C}_q[B^-]^\times$.*

PROOF. For $\varphi, \psi \in \mathbb{C}_q[G/N^-]$, $u \in U^{\leq 0}$ we have

$$\begin{aligned} \langle \Xi'_w(\varphi\psi), u \rangle &= \langle \dot{T}_w(\varphi\psi), Su \rangle = \langle (\dot{T}_w\varphi)(\dot{T}_w\psi), Su \rangle \\ &= \sum_{(u)} \langle \dot{T}_w\varphi, Su_{(1)} \rangle \langle \dot{T}_w\psi, Su_{(0)} \rangle = \sum_{(u)} \langle \Xi'_w(\varphi), u_{(1)} \rangle \langle \Xi'_w(\psi), u_{(0)} \rangle \\ &= \langle \Xi'_w(\psi) \Xi'_w(\varphi), u \rangle. \end{aligned}$$

Here, the second equality is a consequence of Corollary 2.7. For $\lambda \in P^-$, $t \in U^0$, $y \in U^{\leq 0}$ we have

$$\begin{aligned} \langle \Xi'_w(\sigma_\lambda^w), ty \rangle &= \langle \dot{T}_w\sigma_\lambda^w, (Sy)(St) \rangle = \langle v_\lambda^* \dot{T}_w^{-1}, ((Sy)(St)) \dot{T}_w v_\lambda \rangle \\ &= \langle v_\lambda^*, \{ \dot{T}_w^{-1}(Sy) \} \{ \dot{T}_w^{-1}(St) \} v_\lambda \rangle = \varepsilon(\dot{T}_w^{-1}(Sy)) \langle \chi_\lambda, \dot{T}_w^{-1}(St) \rangle \\ &= \varepsilon(y) \langle \chi_{-w\lambda}, t \rangle = \langle \chi_{-w\lambda}, ty \rangle. \end{aligned}$$

□

Hence Ξ'_w induces an algebra anti-homomorphism

$$(4.12) \quad \Xi''_w : \mathcal{S}_w^{-1} \mathbb{C}_q[G/N^-] \rightarrow \mathbb{C}_q[B^-].$$

LEMMA 4.9. *For $\varphi \in \mathcal{S}_w^{-1}(\mathbb{C}_q[G/N^-] \cap \mathbb{C}_q[N_w^- \setminus G])$ we have $\Xi''_w(\varphi) = Sw(\eta_w(\varphi))$.*

PROOF. We may assume $\varphi \in \mathbb{C}_q[G/N^-] \cap \mathbb{C}_q[N_w^- \setminus G]$. By (2.15) the multiplication of $U^{\leq 0}$ induces an isomorphism

$$U^{\leq 0} \cong S^{-1}(U^- \cap \dot{T}_w(U^-)) \otimes U^0 \otimes S^{-1}(U^-[\dot{T}_w])$$

of vector spaces. Let $y_1 \in U^-[\dot{T}_w]$, $y_2 \in U^- \cap \dot{T}_w(U^-)$, $t \in U^0$. Then we have

$$\begin{aligned} \langle \Xi'_w(\varphi), (S^{-1}y_2)t(S^{-1}y_1) \rangle &= \langle \dot{T}_w\varphi, y_1(St)y_2 \rangle = \langle y_2 \dot{T}_w\varphi y_1, St \rangle \\ &= \langle \dot{T}_w(\dot{T}_w^{-1}(y_2))\varphi y_1, St \rangle = \varepsilon(y_1)\varepsilon(y_2)\langle \dot{T}_w\varphi, St \rangle \\ &= \varepsilon(y_1)\varepsilon(y_2)\langle \varphi \dot{T}_w, \dot{T}_w^{-1}S(t) \rangle = \langle \eta_w(\varphi), \dot{T}_w^{-1}(St) \rangle \varepsilon(y_1)\varepsilon(y_2) \\ &= \langle Sw(\eta_w(\varphi)), (S^{-1}y_2)t(S^{-1}y_1) \rangle. \end{aligned}$$

□

By Proposition 4.7, Lemma 4.8 and Lemma 4.9 we obtain a homomorphism

$$(4.13) \quad \Xi_w : \mathcal{M}_w \rightarrow \mathbb{C}_q[B^-]^{\bullet w}$$

of right $\mathbb{C}_q[H]$ -modules given by

$$\Xi_w(\varphi \star \chi) = \Xi''_w(\varphi) \bullet_w \chi \quad (\varphi \in \mathcal{S}_w^{-1} \mathbb{C}_q[G/N^-], \chi \in \mathbb{C}_q[H]).$$

Since Ξ''_w is an algebra anti-homomorphism, we have

$$(4.14) \quad \Xi_w(\varphi\psi \star 1) = \Xi_w(\psi \star 1) \Xi_w(\varphi \star 1) \quad (\varphi, \psi \in \mathcal{S}_w^{-1} \mathbb{C}_q[G/N^-]).$$

4.6. By (2.17) and Lemma 2.9 we can define an injective linear map

$$(4.15) \quad \Omega_w : U^{\geq 0}[\dot{T}_w^{-1}]^\star \rightarrow \mathbb{C}_q[B^-]^{\bullet w}$$

by

$$\begin{aligned} \langle \Omega_w(f), ty_2y_1 \rangle &= \varepsilon(y_2) \langle f, \dot{T}_w^{-1} S(ty_1) \rangle \\ &\quad (f \in U^{\geq 0}[\dot{T}_w^{-1}]^\star, y_1 \in U^-[\hat{T}_w], y_2 \in U^- \cap \hat{T}_w U^-, t \in U^0). \end{aligned}$$

LEMMA 4.10. *The linear map Ω_w is a homomorphism of right $\mathbb{C}_q[H]$ -modules.*

PROOF. Let $f \in U^{\geq 0}[\dot{T}_w^{-1}]^\star$ and $\chi \in \mathbb{C}_q[H]$. For $y_1 \in U^-[\hat{T}_w]$, $y_2 \in U^- \cap \hat{T}_w U^-$, $t \in U^0$ we have

$$\begin{aligned} \langle \Omega_w(f) \bullet_w \chi, ty_2y_1 \rangle &= \langle (Sw\chi) \Omega_w(f), ty_2y_1 \rangle \\ &= \sum_{(y_1), (y_2), (t)} \langle Sw\chi, t_{(0)}y_{2(0)}y_{1(0)} \rangle \langle \Omega_w(f), t_{(1)}y_{2(1)}y_{1(1)} \rangle \\ &= \sum_{(t)} \langle Sw\chi, t_{(0)} \rangle \langle \Omega_w(f), t_{(1)}y_2y_1 \rangle \\ &= \sum_{(t)} \varepsilon(y_2) \langle Sw\chi, t_{(0)} \rangle \langle f, \dot{T}_w^{-1} S(t_{(1)}y_1) \rangle \\ &= \sum_{(t)} \varepsilon(y_2) \langle \chi, \dot{T}_w^{-1} S(t_{(0)}) \rangle \langle f, \{\dot{T}_w^{-1} S(y_1)\} \{\dot{T}_w^{-1} S(t_{(1)})\} \rangle \\ &= \varepsilon(y_2) \langle f\chi, \{\dot{T}_w^{-1} S(y_1)\} \{\dot{T}_w^{-1} S(t)\} \rangle \\ &= \varepsilon(y_2) \langle f\chi, \dot{T}_w^{-1} S(ty_1) \rangle \\ &= \langle \Omega_w(f\chi), ty_2y_1 \rangle. \end{aligned}$$

□

LEMMA 4.11. $\Omega_w \circ \Theta_w = \Xi_w$.

PROOF. By Proposition 4.7 we have only to show

$$(\Omega_w \circ \Theta_w)(\varphi \star \chi) = \Xi_w(\varphi \star \chi) \quad (\varphi \in \mathbb{C}_q[G/N^-], \chi \in \mathbb{C}_q[H]).$$

By the definitions of Θ_w , Ξ_w and Lemma 4.10 it is sufficient to show

$$(\Omega_w \circ \Theta'_w)(\varphi) = \Xi'_w(\varphi) \quad (\varphi \in \mathbb{C}_q[G/N^-]).$$

Let $y_1 \in U^-[\hat{T}_w]$, $y_2 \in U_{-\gamma}^- \cap \hat{T}_w U^-$, $\delta \in Q$. Then we have

$$\begin{aligned} \langle (\Omega_w \circ \Theta'_w)(\varphi), k_\delta y_2 y_1 \rangle &= \varepsilon(y_2) \langle \Theta'_w(\varphi), \dot{T}_w^{-1} S(k_\delta y_1) \rangle \\ &= \varepsilon(y_2) \langle \varphi \dot{T}_w, \dot{T}_w^{-1} S(k_\delta y_1) \rangle = \varepsilon(y_2) \langle \dot{T}_w \varphi, S(k_\delta y_1) \rangle, \\ \langle \Xi'_w(\varphi), k_\delta y_2 y_1 \rangle &= \langle \dot{T}_w \varphi, (Sy_1)(Sy_2)k_{-\delta} \rangle \\ &= q^{-(\gamma, \delta)} \langle \dot{T}_w \varphi, (Sy_1)k_{-\delta}(Sy_2) \rangle = q^{-(\gamma, \delta)} \langle (Sy_2) \dot{T}_w \varphi, (Sy_1)k_{-\delta} \rangle \\ &= q^{-(\gamma, \delta)} \langle \dot{T}_w(\dot{T}_w^{-1} S(y_2)) \varphi, S(k_\delta y_1) \rangle \\ &= q^{-(\gamma, \delta)} \langle \dot{T}_w(S\hat{T}_w^{-1}(y_2)) \varphi, S(k_\delta y_1) \rangle = \varepsilon(y_2) \langle \dot{T}_w \varphi, S(k_\delta y_1) \rangle. \end{aligned}$$

□

We define a $\mathbb{C}_q[H]$ -submodule \mathcal{A}_w of $\mathbb{C}_q[B^-]^{\bullet w}$ by

$$\mathcal{A}_w = \{\varphi \in \mathbb{C}_q[B^-] \mid \varphi y = 0 \quad (y \in U^- \cap \hat{T}_w(U^-))\}.$$

Note that we have an isomorphism

$$(4.16) \quad (U^-)^{\star} \otimes \mathbb{C}_q[H] \cong \mathbb{C}_q[B^-]^{\bullet w} \quad (f \otimes \chi \leftrightarrow d_{f \otimes \chi})$$

of right $\mathbb{C}_q[H]$ -modules given by

$$\langle d_{f \otimes \chi}, ty \rangle = \langle Sw\chi, t \rangle \langle f, y \rangle \quad (t \in U^0, y \in U^-).$$

Set

$$(4.17) \quad U^-[\hat{T}_w]^{\star} = \bigoplus_{\gamma \in Q^+} (U^-[\hat{T}_w] \cap U_{-\gamma}^-)^* \subset (U^-[\hat{T}_w])^*.$$

By (2.17) we have an injective linear map

$$(4.18) \quad i_w^- : U^-[\hat{T}_w]^{\star} \rightarrow (U^-)^{\star}$$

given by

$$\langle i_w^-(f), y'y \rangle = \varepsilon(y') \langle f, y \rangle \quad (y \in U^-[\hat{T}_w], y' \in U^- \cap \hat{T}_w(U^-))$$

Under the identification (4.16) we have

$$i_w^-(U^-[\hat{T}_w]^{\star}) \otimes \mathbb{C}_q[H] \cong \mathcal{A}_w.$$

PROPOSITION 4.12. *The linear map Ξ_w is injective and its image coincides with \mathcal{A}_w .*

PROOF. Note that Θ_w is bijective and Ω_w is injective. Hence by Lemma 4.11 we see that Ξ_w is injective and its image coincides with $\text{Im}(\Omega_w)$. Moreover, by the definition of Ω_w the image of Ω_w coincides with \mathcal{A}_w . \square

5. THE DECOMPOSITION INTO TENSOR PRODUCT

5.1. For $i \in I$ define a Hopf subalgebra $U(i)$ of U by

$$U(i) = \langle k_i^{\pm 1}, e_i, f_i \rangle \cong \mathbb{F} \otimes_{\mathbb{Q}(q_i)} U_{q_i}(\mathfrak{sl}_2) \subset U.$$

Define subalgebras $U(i)^b$ ($b = 0, \pm, \geq 0, \leq 0$) by

$$\begin{aligned} U(i)^0 &= \langle k_i^{\pm 1} \rangle, & U(i)^+ &= \langle e_i \rangle, & U(i)^- &= \langle f_i \rangle, \\ U(i)^{\geq 0} &= \langle k_i^{\pm 1}, e_i \rangle, & U(i)^{\leq 0} &= \langle k_i^{\pm 1}, f_i \rangle. \end{aligned}$$

We denote the quantized coordinate algebra of $U(i)$ by $\mathbb{C}_q[G(i)]$ ($\cong \mathbb{F} \otimes_{\mathbb{Q}(q_i)} \mathbb{C}_{q_i}[SL_2]$). As an algebra it is generated by elements a_i, b_i, c_i, d_i satisfying the fundamental relations

$$\begin{aligned} a_i b_i &= q_i b_i a_i, & c_i d_i &= q_i d_i c_i, & a_i c_i &= q_i c_i a_i, & b_i d_i &= q_i d_i b_i, \\ b_i c_i &= c_i b_i, & a_i d_i - d_i a_i &= (q_i - q_i^{-1}) b_i c_i, & a_i d_i - q_i b_i c_i &= 1 \end{aligned}$$

(see Example 3.1).

We have a quotient Hopf algebra $\mathbb{C}_q[H(i)]$ of $\mathbb{C}_q[G(i)]$ corresponding to $U(i)^0$. Then we have

$$\mathbb{C}_q[H(i)] = \mathbb{F}[\chi_i^{\pm 1}], \quad \chi_i(k_i) = q_i.$$

Denote by

$$(5.1) \quad r_{G(i)}^G : \mathbb{C}_q[G] \rightarrow \mathbb{C}_q[G(i)]$$

the Hopf algebra homomorphism corresponding to $U(i) \subset U$.

5.2. Consider the $(\mathbb{C}_q[G(i)], \mathbb{C}_q[H(i)])$ -bimodule

$$(5.2) \quad \mathcal{M}_i = \mathbb{F} \otimes_{\mathbb{Q}(q_i)} \mathcal{M}_{s_i}^{SL_2},$$

where s_i is the generator of the Weyl group of $U(i)$, and $\mathcal{M}_{s_i}^{SL_2}$ is the $\mathbb{C}_q[G]$ -module \mathcal{M}_w for $G = SL_2$, $q = q_i$, $w = s_i$. We have an isomorphism

$$\Theta_i : \mathcal{M}_i \rightarrow (U(i)^{\geq 0})^\star$$

of right $\mathbb{C}_q[H(i)]$ -modules given by

$$\begin{aligned} \langle \Theta_i(\varphi \star \chi), xt \rangle &= \sum_{(t)} \langle \varphi \dot{T}_i, xt_{(0)} \rangle \langle \chi, t_{(1)} \rangle \\ &(\varphi \in \mathbb{C}_q[G(i)], \chi \in \mathbb{C}_q[H(i)], x \in U(i)^+, t \in U(i)^0). \end{aligned}$$

Define $p_i(n) \in \mathcal{M}_i$ by

$$\langle \Theta_i(p_i(n)), e_i^{n'} k_i^j \rangle = \delta_{nn'} (-1)^n q_i^n [n]_{q_i}!.$$

By Lemma 4.6 we obtain the following.

PROPOSITION 5.1. *The set $\{p_i(n) \mid n \in \mathbb{Z}_{\geq 0}\}$ forms a basis of the $\mathbb{C}_q[H(i)]$ -module \mathcal{M}_i . Moreover, we have*

$$\begin{aligned} a_i p_i(n) &= (1 - q_i^{2n}) \chi_i p_i(n-1), & b_i p_i(n) &= \chi_i^{-1} q_i^n p_i(n), \\ c_i p_i(n) &= -\chi_i q_i^{n+1} p_i(n), & d_i p_i(n) &= \chi_i^{-1} p_i(n+1). \end{aligned}$$

We will regard \mathcal{M}_i as a $(\mathbb{C}_q[G], \mathbb{C}_q[H(i)])$ -module via $r_{G(i)}^G$.

5.3. Let $w \in W$ with $\ell(w) = m$. We set

$$(5.3) \quad z_{\mathbf{i}, r} = s_{i_{r+1}} s_{i_{r+2}} \cdots s_{i_m} \quad (r = 0, \dots, m),$$

$$(5.4) \quad \mathbb{C}_q[H(\mathbf{i})] = \mathbb{C}_q[H(i_1)] \otimes \cdots \otimes \mathbb{C}_q[H(i_t)].$$

For $\mathbf{i} \in \mathcal{I}_w$ consider the $(\mathbb{C}_q[G]^{\otimes t}, \mathbb{C}_q[H(\mathbf{i})])$ -bimodule $\mathcal{M}_{i_1} \otimes \cdots \otimes \mathcal{M}_{i_m}$. Via the iterated comultiplication $\Delta_{t-1} : \mathbb{C}_q[G] \rightarrow \mathbb{C}_q[G]^{\otimes t}$ and the algebra homomorphism

$$(5.5) \quad \Delta_{\mathbf{i}} : \mathbb{C}_q[H] \rightarrow \mathbb{C}_q[H(\mathbf{i})]$$

given by

$$\Delta_{\mathbf{i}}(\chi) = \sum_{(\chi)_{m-1}} z_{\mathbf{i}, 1} \chi_{(0)}|_{U(i_1)^0} \otimes \cdots \otimes z_{\mathbf{i}, m} \chi_{(m-1)}|_{U(i_m)^0},$$

we can regard $\mathcal{M}_{i_1} \otimes \cdots \otimes \mathcal{M}_{i_m}$ as a $(\mathbb{C}_q[G], \mathbb{C}_q[H(\mathbf{i})])$ -bimodule or a $(\mathbb{C}_q[G], \mathbb{C}_q[H])$ -bimodule.

Define a linear map

$$F'_{\mathbf{i}} : \mathcal{M}_w \rightarrow \mathcal{M}_{i_1} \otimes \cdots \otimes \mathcal{M}_{i_m}$$

by

$$\begin{aligned} &\langle (\Theta_{i_1} \otimes \cdots \otimes \Theta_{i_m})(F'_{\mathbf{i}}(m)), u_1 \otimes \cdots \otimes u_m \rangle \\ &= \langle \Theta_w(m), (\dot{T}_{z_{i_1, 1}}^{-1}(u_1)) \cdots (\dot{T}_{z_{i_m, m}}^{-1}(u_m)) \rangle \\ &\quad (m \in \mathcal{M}_w, u_j \in U(i_j)^{\geq 0}). \end{aligned}$$

In this subsection we will show the following.

THEOREM 5.2. *The linear map F'_i is a homomorphism of $(\mathbb{C}_q[G], \mathbb{C}_q[H])$ -bimodules, and it induces an isomorphism*

$$(5.6) \quad F_i : \mathcal{M}_w \otimes_{\mathbb{C}_q[H]} \mathbb{C}_q[H(\mathbf{i})] \rightarrow \mathcal{M}_{i_1} \otimes \cdots \otimes \mathcal{M}_{i_m}$$

of $(\mathbb{C}_q[G], \mathbb{C}_q[H(\mathbf{i})])$ -bimodules, where $\mathbb{C}_q[H] \rightarrow \mathbb{C}_q[H(\mathbf{i})]$ is given by Δ_i .

We first note the following.

LEMMA 5.3. *Let $\varphi \in \mathbb{C}_q[G]$, $w, w_1, \dots, w_k \in W$. Then we have*

$$\begin{aligned} & \Delta_k(\varphi \dot{T}_w) \\ &= \sum_{(\varphi)_k} (\dot{T}_{w_1}^{-1} \varphi_{(0)} \dot{T}_w) \otimes (\dot{T}_{w_2}^{-1} \varphi_{(1)} \dot{T}_{w_1}) \otimes \cdots \otimes (\dot{T}_{w_k}^{-1} \varphi_{(k-1)} \dot{T}_{w_{k-1}}) \otimes (\varphi_{(k)} \dot{T}_{w_k}). \end{aligned}$$

PROOF. By induction we may assume that $k = 1$. Set $x = w_1$. We may also assume that $\varphi = \Phi_{v^* \otimes v}$ ($v \in V, v^* \in V^*$) for some $V \in \text{Mod}_0(U)$. Let $\{v_j\}$ be a basis of V and let $\{v_j^*\}$ be its dual basis. Then we have

$$\Delta(\Phi_{v^* \otimes v}) = \sum_j \Phi_{v^* \otimes v_j} \otimes \Phi_{v_j^* \otimes v}.$$

Since the dual basis of $\{\dot{T}_x^{-1} v_j\}$ is $\{v_j^* \dot{T}_x\}$, we have for $u_0, u_1 \in U$ that

$$\begin{aligned} & \langle \Delta(\Phi_{v^* \otimes v} \dot{T}_w), u_0 \otimes u_1 \rangle = \langle v^* \dot{T}_w, u_0 u_1 v \rangle \\ &= \sum_j \langle v_j^* \dot{T}_x, u_1 v \rangle \langle v^* \dot{T}_w, u_0 \dot{T}_x^{-1} v_j \rangle = \sum_j \langle \Phi_{v^* \dot{T}_w \otimes \dot{T}_x^{-1} v_j} \otimes \Phi_{v_j^* \dot{T}_x \otimes v}, u_0 \otimes u_1 \rangle \\ &= \sum_j \langle (\dot{T}_x^{-1} \Phi_{v^* \otimes v_j} \dot{T}_w) \otimes (\Phi_{v_j^* \otimes v} \dot{T}_x), u_0 \otimes u_1 \rangle \end{aligned}$$

□

LEMMA 5.4. *For $\varphi \in \mathbb{C}_q[G]$ we have*

$$\begin{aligned} & \langle \Theta_w(\varphi \star 1), (\dot{T}_{z_{i,1}}^{-1}(u_1)) \cdots (\dot{T}_{z_{i,m}}^{-1}(u_m)) \rangle \\ &= \sum_{(\varphi)_{m-1}} \prod_{r=1}^m \langle \Theta_{i_r}(r_{G(i_r)}^G(\varphi_{(r-1)}) \star 1), u_r \rangle \\ & \quad (u_r \in U(i_r)^{\geq 0}). \end{aligned}$$

PROOF. By Lemma 5.3 we have

$$\begin{aligned} & \Delta_{m-1}(\varphi \dot{T}_w) \\ &= \sum_{(\varphi)_m} (\dot{T}_{z_{i,1}}^{-1} \varphi_{(0)} \dot{T}_{z_{i,0}}) \otimes (\dot{T}_{z_{i,2}}^{-1} \varphi_{(1)} \dot{T}_{z_{i,1}}) \otimes \cdots \otimes (\dot{T}_{z_{i,m}}^{-1} \varphi_{(m-1)} \dot{T}_{z_{i,m-1}}). \end{aligned}$$

Hence

$$\begin{aligned}
& \langle \Theta_w(\varphi \star 1), (\dot{T}_{z_{i,1}}^{-1}(u_1)) \cdots (\dot{T}_{z_{i,m}}^{-1}(u_m)) \rangle \\
&= \langle \varphi \dot{T}_w, (\dot{T}_{z_{i,1}}^{-1}(u_1)) \cdots (\dot{T}_{z_{i,m}}^{-1}(u_m)) \rangle \\
&= \langle \Delta_{m-1}(\varphi \dot{T}_w), (\dot{T}_{z_{i,1}}^{-1}(u_1)) \otimes \cdots \otimes (\dot{T}_{z_{i,m}}^{-1}(u_m)) \rangle \\
&= \sum_{(\varphi)_{m-1}} \prod_{r=1}^m \langle \dot{T}_{z_{i,r}}^{-1} \varphi_{(r-1)} \dot{T}_{z_{i,r-1}}, \dot{T}_{z_{i,r}}^{-1}(u_r) \rangle \\
&= \sum_{(\varphi)_{m-1}} \langle \prod_{r=1}^m \varphi_{(r-1)} \dot{T}_{i_r}, u_r \rangle \\
&= \sum_{(\varphi)_{m-1}} \prod_{r=1}^m \langle \Theta_{i_r}(r_{G(i_r)}^G(\varphi_{(r-1)}) \star 1), u_r \rangle
\end{aligned}$$

by Lemma 3.16. □

Now we give a proof of Theorem 5.2.

We first show that F'_i is a homomorphism of right $\mathbb{C}_q[H]$ -modules. For $m \in \mathcal{M}_w$, $x_j \in U(i_j)_{c_j \alpha_{i_j}}^+$, $\chi \in \mathbb{C}_q[H]$ we have

$$\begin{aligned}
& \langle (\Theta_{i_1} \otimes \cdots \otimes \Theta_{i_m})(F'_i(m\chi)), x_1 k_{i_1}^{p_1} \otimes \cdots \otimes x_m k_{i_m}^{p_m} \rangle \\
&= \langle \Theta_w(m\chi), (\dot{T}_{z_{i,1}}^{-1}(x_1 k_{i_1}^{p_1})) \cdots (\dot{T}_{z_{i,m}}^{-1}(x_m k_{i_m}^{p_m})) \rangle \\
&= q^A \langle \Theta_w(m\chi), (\dot{T}_{z_{i,1}}^{-1}(x_1)) \cdots (\dot{T}_{z_{i,m}}^{-1}(x_m)) k_{z_{i,1}}^{p_1} \cdots k_{z_{i,m}}^{p_m} \rangle \\
&= q^A \langle \Theta_w(m), (\dot{T}_{z_{i,1}}^{-1}(x_1)) \cdots (\dot{T}_{z_{i,m}}^{-1}(x_m)) k_{z_{i,1}}^{p_1} \cdots k_{z_{i,m}}^{p_m} \rangle \\
&\quad \times \langle \chi, k_{z_{i,1}}^{p_1} \cdots k_{z_{i,m}}^{p_m} \rangle \\
&= \langle \Theta_w(m), (\dot{T}_{z_{i,1}}^{-1}(x_1 k_{i_1}^{p_1})) \cdots (\dot{T}_{z_{i,m}}^{-1}(x_m k_{i_m}^{p_m})) \rangle \langle \chi, k_{z_{i,1}}^{p_1} \cdots k_{z_{i,m}}^{p_m} \rangle \\
&= \langle (\Theta_{i_1} \otimes \cdots \otimes \Theta_{i_m})(F'_i(m)), x_1 k_{i_1}^{p_1} \otimes \cdots \otimes x_m k_{i_m}^{p_m} \rangle \\
&\quad \times \langle \chi, k_{z_{i,1}}^{p_1} \cdots k_{z_{i,m}}^{p_m} \rangle \\
&= \langle \{(\Theta_{i_1} \otimes \cdots \otimes \Theta_{i_m})(F'_i(m))\} \Delta_i(\chi), x_1 k_{i_1}^{p_1} \otimes \cdots \otimes x_m k_{i_m}^{p_m} \rangle \\
&= \langle (\Theta_{i_1} \otimes \cdots \otimes \Theta_{i_m})(F'_i(m) \Delta_i(\chi)), x_1 k_{i_1}^{p_1} \otimes \cdots \otimes x_m k_{i_m}^{p_m} \rangle,
\end{aligned}$$

where

$$A = \sum_{r=1}^{m-1} p_r(z_{i,r}^{-1} \alpha_{i_r}, c_{r+1} z_{i,r+1}^{-1} \alpha_{i_{r+1}} + \cdots + c_m z_{i,m}^{-1} \alpha_{i_m}).$$

Hence F'_i is a homomorphism of right $\mathbb{C}_q[H]$ -modules.

We next show that F'_i is a homomorphism of left $\mathbb{C}_q[G]$ -modules. It is sufficient to show $F'_i(\varphi m) = \varphi F'_i(m)$ for $\varphi \in \mathbb{C}_q[G]$, $m \in \mathcal{M}_w$. Since F'_i is a homomorphism of right $\mathbb{C}_q[H]$ -module, we may assume that $m = \psi \star 1$ ($\psi \in$

$\mathbb{C}_q[G]$. Then we have

$$\begin{aligned}
& \langle (\Theta_{i_1} \otimes \cdots \otimes \Theta_{i_m})(F'_i(\psi \star 1)), u_1 \otimes \cdots \otimes u_m \rangle \\
&= \langle \Theta_w(\psi \star 1), (\dot{T}_{z_{i,1}}^{-1}(u_1)) \cdots (\dot{T}_{z_{i,m}}^{-1}(u_m)) \rangle \\
&= \sum_{(\psi)_{m-1}} \prod_{r=1}^m \langle \Theta_{i_r}(r_{G(i_r)}^G(\psi_{(r-1)}) \star 1), u_r \rangle \\
&= \sum_{(\psi)_{m-1}} \langle (\Theta_{i_1} \otimes \cdots \otimes \Theta_{i_m})((r_{G(i_1)}^G(\psi_{(0)}) \star 1) \otimes \cdots \otimes (r_{G(i_{m-1})}^G(\psi_{(m-1)}) \star 1)) \\
&\quad, u_1 \otimes \cdots \otimes u_m \rangle.
\end{aligned}$$

Hence

$$F'_i(\psi \star 1) = \sum_{(\psi)_{m-1}} (r_{G(i_1)}^G(\psi_{(0)}) \star 1) \otimes \cdots \otimes (r_{G(i_m)}^G(\psi_{(m-1)}) \star 1).$$

It follows that

$$\begin{aligned}
& F'_i(\varphi m) = F'_i(\varphi \psi \star 1) \\
&= \sum_{(\varphi)_{m-1}, (\psi)_{m-1}} (r_{G(i_1)}^G(\varphi_{(0)} \psi_{(0)}) \star 1) \otimes \cdots \otimes (r_{G(i_m)}^G(\varphi_{(m-1)} \psi_{(m-1)}) \star 1) \\
&= \varphi \sum_{(\psi)_{m-1}} (r_{G(i_1)}^G(\psi_{(0)}) \star 1) \otimes \cdots \otimes (r_{G(i_m)}^G(\psi_{(m-1)}) \star 1) = \varphi F'_i(m).
\end{aligned}$$

Therefore, F'_i is a homomorphism of left $\mathbb{C}_q[G]$ -modules.

Since F'_i is a homomorphism of $(\mathbb{C}_q[G], \mathbb{C}_q[H])$ -bimodules, it induces a homomorphism

$$F_i : \mathcal{M}_w \otimes_{\mathbb{C}_q[H]} \mathbb{C}_q[H(\mathbf{i})] \rightarrow \mathcal{M}_{i_1} \otimes \cdots \otimes \mathcal{M}_{i_m} \quad (a \otimes \chi \mapsto F'_i(a)\chi)$$

of $(\mathbb{C}_q[G], \mathbb{C}_q[H(\mathbf{i})])$ -bimodules. It remains to show that F_i is bijective. Via Θ_w we have

$$\begin{aligned}
\mathcal{M}_w \otimes_{\mathbb{C}_q[H]} \mathbb{C}_q[H(\mathbf{i})] &\cong (U^+[\dot{T}_w^{-1}]^\star \otimes \mathbb{C}_q[H]) \otimes_{\mathbb{C}_q[H]} \mathbb{C}_q[H(\mathbf{i})] \\
&\cong U^+[\dot{T}_w^{-1}]^\star \otimes \mathbb{C}_q[H(\mathbf{i})],
\end{aligned}$$

and via $\Theta_{i_1} \otimes \cdots \otimes \Theta_{i_m}$ we have

$$\begin{aligned}
& \mathcal{M}_{i_1} \otimes \cdots \otimes \mathcal{M}_{i_m} \\
&\cong \{(U(i_1)^+)^\star \otimes \mathbb{C}_q[H(i_1)]\} \otimes \cdots \otimes \{(U(i_m)^+)^\star \otimes \mathbb{C}_q[H(i_m)]\} \\
&\cong \{(U(i_1)^+)^\star \otimes \cdots \otimes (U(i_m)^+)^\star\} \otimes \mathbb{C}_q[H(\mathbf{i})].
\end{aligned}$$

Hence the assertion follows from

$$\begin{aligned}
(U(i_1)^+)^\star \otimes \cdots \otimes (U(i_m)^+)^\star &\cong U^+[\dot{T}_w^{-1}]^\star \\
(x_1 \otimes \cdots \otimes x_m &\leftrightarrow \dot{T}_{z_{i,1}}^{-1}(x_1) \otimes \cdots \otimes \dot{T}_{z_{i,m}}^{-1}(x_m)).
\end{aligned}$$

The proof of Theorem 5.2 is complete

6. BASIS ELEMENTS

6.1. Let $w \in W$ with $\ell(w) = m$, and fix $\mathbf{i} = (i_1, \dots, i_m) \in \mathcal{I}_w$.

For $\mathbf{n} = (n_1, \dots, n_m) \in (\mathbb{Z}_{\geq 0})^m$ we denote by $p_{\mathbf{i}}(\mathbf{n})$ the element of $\mathcal{M}_w \otimes_{\mathbb{C}_q[H]} \mathbb{C}_q[H(\mathbf{i})]$ corresponding to

$$p_{i_1}(n_1) \otimes \dots \otimes p_{i_m}(n_m) \in \mathcal{M}_{i_1} \otimes \dots \otimes \mathcal{M}_{i_m}$$

under the isomorphism (5.6).

By (4.7) we have

$$U^{\geq 0}[\dot{T}_w^{-1}]^{\star} \otimes_{\mathbb{C}_q[H]} \mathbb{C}_q[H(\mathbf{i})] \cong U^+[\dot{T}_w^{-1}]^{\star} \otimes \mathbb{C}_q[H(\mathbf{i})].$$

Hence Θ_w induces an isomorphism

$$(6.1) \quad \Theta_{w,\mathbf{i}} : \mathcal{M}_w \otimes_{\mathbb{C}_q[H]} \mathbb{C}_q[H(\mathbf{i})] \rightarrow U^+[\dot{T}_w^{-1}]^{\star} \otimes \mathbb{C}_q[H(\mathbf{i})]$$

of right $\mathbb{C}_q[H(\mathbf{i})]$ -modules. We will regard $U^+[\dot{T}_w^{-1}]^{\star} \otimes \mathbb{C}_q[H(\mathbf{i})]$ as a subset of $\text{Hom}_{\mathbb{F}}(U^+[\dot{T}_w^{-1}], \mathbb{C}_q[H(\mathbf{i})])$ in the following.

For $r = 1, \dots, m$ and $\mathbf{n} \in (\mathbb{Z}_{\geq 0})^m$ set

$$(6.2) \quad \beta_{\mathbf{i},r} = z_{\mathbf{i},r}^{-1} \alpha_{i_r}, \quad \gamma_{\mathbf{i},\mathbf{n},r} = n_{r+1} \beta_{\mathbf{i},r+1} + \dots + n_m \beta_{\mathbf{i},m}.$$

PROPOSITION 6.1.

$$\begin{aligned} & \langle \Theta_{w,\mathbf{i}}(p_{\mathbf{i}}(\mathbf{n})), \tilde{e}_{\mathbf{i}}^{\mathbf{n}'} \rangle \\ &= \delta_{\mathbf{n},\mathbf{n}'} \left\{ \prod_{r=1}^m (-1)^{n_r} q_{i_r}^{n_r} [n_r]_{q_{i_r}}! \right\} \chi_{i_1}^{\langle \beta_{\mathbf{i},1}^{\vee}, \gamma_{\mathbf{i},\mathbf{n},1} \rangle} \otimes \dots \otimes \chi_{i_m}^{\langle \beta_{\mathbf{i},m}^{\vee}, \gamma_{\mathbf{i},\mathbf{n},m} \rangle}. \end{aligned}$$

PROOF. Define $a \in \mathcal{M}_w$ by

$$\langle \Theta_w(a), \tilde{e}_{\mathbf{i}}^{\mathbf{n}'} t \rangle = \delta_{\mathbf{n}\mathbf{n}'} \varepsilon(t) \quad (\mathbf{n}' \in (\mathbb{Z}_{\geq 0})^m, t \in U^0).$$

Then we have

$$\begin{aligned} & \langle (\Theta_{i_1} \otimes \dots \otimes \Theta_{i_m})(F_{\mathbf{i}}(a \otimes 1)), e_{i_1}^{n'_1} k_{i_1}^{j_1} \otimes \dots \otimes e_{i_m}^{n'_m} k_{i_m}^{j_m} \rangle \\ &= \langle (\Theta_{i_1} \otimes \dots \otimes \Theta_{i_m})(F'_{\mathbf{i}}(a)), e_{i_1}^{n'_1} k_{i_1}^{j_1} \otimes \dots \otimes e_{i_m}^{n'_m} k_{i_m}^{j_m} \rangle \\ &= \langle \Theta_w(a), T_{z_{i_1,1}}^{-1}(e_{i_1}^{n'_1} k_{i_1}^{j_1}) \dots T_{z_{i_m,m}}^{-1}(e_{i_m}^{n'_m} k_{i_m}^{j_m}) \rangle \\ &= q^{A'} \langle \Theta_w(a), \tilde{e}_{\mathbf{i}}^{\mathbf{n}'} k_{z_{i_1,1}^{-1} \alpha_{i_1}}^{j_1} \dots k_{z_{i_m,m}^{-1} \alpha_{i_m}}^{j_m} \rangle \\ &= \delta_{\mathbf{n}\mathbf{n}'} q^A \\ &= \delta_{\mathbf{n}\mathbf{n}'} \prod_{r=1}^m (q_{i_r}^{j_r})^{\langle \beta_{\mathbf{i},r}^{\vee}, \gamma_{\mathbf{i},\mathbf{n},r} \rangle}, \end{aligned}$$

where

$$A' = \sum_{r=1}^{m-1} \langle j_r \beta_{\mathbf{i},r}, \gamma_{\mathbf{i},\mathbf{n},r+1} \rangle, \quad A = \sum_{r=1}^{m-1} \langle j_r \beta_{\mathbf{i},r}, \gamma_{\mathbf{i},\mathbf{n}',r+1} \rangle.$$

On the other hand we have

$$\begin{aligned} & \langle (\Theta_{i_1} \otimes \dots \otimes \Theta_{i_m})(p_{i_1}(n_1) \otimes \dots \otimes p_{i_m}(n_m)), e_{i_1}^{n'_1} k_{i_1}^{j_1} \otimes \dots \otimes e_{i_m}^{n'_m} k_{i_m}^{j_m} \rangle \\ &= \prod_{r=1}^m \delta_{n_r n'_r} (-1)^{n_r} q_{i_r}^{n_r} [n_r]_{q_{i_r}}! = \delta_{\mathbf{n}\mathbf{n}'} \prod_{r=1}^m (-1)^{n_r} q_{i_r}^{n_r} [n_r]_{q_{i_r}}!. \end{aligned}$$

□

6.2. We rewrite Proposition 6.1 using Ξ_w instead of Θ_w . Note that the isomorphism (4.16) induces

$$\mathbb{C}_q[B^-]^{\bullet w} \otimes_{\mathbb{C}_q[H]} \mathbb{C}_q[H(\mathbf{i})] \cong (U^-)^{\star} \otimes \mathbb{C}_q[H(\mathbf{i})] \quad (\subset \text{Hom}_{\mathbb{F}}(U^-, \mathbb{C}_q[H(\mathbf{i})])).$$

Hence Ξ_w induces an injection

$$\Xi_{w,i} : \mathcal{M}_w \otimes_{\mathbb{C}_q[H]} \mathbb{C}_q[H(\mathbf{i})] \rightarrow (U^-)^{\star} \otimes \mathbb{C}_q[H(\mathbf{i})] \quad (\subset \text{Hom}_{\mathbb{F}}(U^-, \mathbb{C}_q[H(\mathbf{i})])).$$

Recall that $\{\hat{f}_{\mathbf{i}}^{\mathbf{n}}\}_{\mathbf{n}}$ forms a basis $U^-[\hat{T}_w]$ and the multiplication induces an isomorphism $(U^- \cap \hat{T}_w U^-) \otimes U^-[\hat{T}_w] \cong U^-$. (see Proposition 2.12, (2.17)).

PROPOSITION 6.2. *For $y \in U^- \cap \hat{T}_w U^-$ we have*

$$\begin{aligned} & \langle \Xi_{w,i}(p_{\mathbf{i}}(\mathbf{n})), y \hat{f}_{\mathbf{i}}^{\mathbf{n}'} \rangle \\ &= \varepsilon(y) \delta_{\mathbf{n}, \mathbf{n}'} \left\{ \prod_{r=1}^m (-1)^{n_r} q_{i_r}^{n_r} [n_r]_{q_{i_r}}! \right\} \chi_{i_1}^{\langle \beta_{i_1,1}^{\vee}, \gamma_1 \rangle} \otimes \cdots \otimes \chi_{i_m}^{\langle \beta_{i_m,m}^{\vee}, \gamma_m \rangle}. \end{aligned}$$

PROOF. Let

$$\Omega_{w,i} : U^+[\dot{T}_w^{-1}]^{\star} \otimes \mathbb{C}_q[H(\mathbf{i})] \rightarrow (U^-)^{\star} \otimes \mathbb{C}_q[H(\mathbf{i})]$$

be the homomorphism of right $\mathbb{C}_q[H(\mathbf{i})]$ -modules induced by Ω_w . For $f \in U^+[\dot{T}_w^{-1}]^{\star}$ the element of $U^{\geq 0}[\dot{T}_w^{-1}]^{\star} \otimes_{\mathbb{C}_q[H]} \mathbb{C}_q[H(\mathbf{i})]$ corresponding to $f \otimes 1 \in U^+[\dot{T}_w^{-1}]^{\star} \otimes \mathbb{C}_q[H(\mathbf{i})]$ is written as $\tilde{f} \otimes 1$, where $\tilde{f} \in U^{\geq 0}[\dot{T}_w^{-1}]^{\star}$ is given by

$$\langle \tilde{f}, xt \rangle = \langle f, x \rangle \varepsilon(t) \quad (x \in U^+[\dot{T}_w^{-1}], t \in U^0).$$

Then for $y_1 \in U^-[\hat{T}_w]$, $y_2 \in U^- \cap \hat{T}_w U^-$, $t \in U^0$ we have

$$\begin{aligned} \langle \Omega_w(\tilde{f}), ty_2 y_1 \rangle &= \varepsilon(y_2) \langle \tilde{f}, \dot{T}_w^{-1} S(ty_1) \rangle = \varepsilon(y_2 t) \langle f, \dot{T}_w^{-1} S(y_1) \rangle \\ & \quad (\dot{T}_w^{-1} S(y_1) \in U^+[\dot{T}_w^{-1}]). \end{aligned}$$

Namely, the element of $(U^-)^{\star} \otimes \mathbb{C}_q[H(\mathbf{i})]$ corresponding to $f \otimes 1$ is written as $\hat{f} \otimes 1$, where $\hat{f} \in (U^-)^{\star}$ is given by

$$\langle \hat{f}, y_2 y_1 \rangle = \varepsilon(y_2) \langle f, \dot{T}_w^{-1} S(y_1) \rangle \quad (y_1 \in U^-[\hat{T}_w], y_2 \in U^- \cap \hat{T}_w U^-).$$

Hence for $y \in U^- \cap \hat{T}_w U^-$ we have

$$\langle \Xi_{w,i}(p_{\mathbf{i}}(\mathbf{n})), y \hat{f}_{\mathbf{i}}^{\mathbf{n}'} \rangle = \varepsilon(y) \langle \Theta_{w,i}(p_{\mathbf{i}}(\mathbf{n})), \tilde{e}_{\mathbf{i}}^{\mathbf{n}'} \rangle.$$

□

6.3. Set

$$(6.3) \quad U^{\geq 0}[\hat{T}_w] = U^+[\hat{T}_w] U^0 \subset U^{\geq 0}$$

and define

$$(6.4) \quad \Psi_w : U^{\geq 0}[\hat{T}_w] \rightarrow \mathbb{C}_q[B^-]$$

by

$$\langle \Psi_w(x), u \rangle = \tau(x, u) \quad (x \in U^{\geq 0}[\hat{T}_w], u \in U^{\leq 0}).$$

By Proposition 2.5 Ψ_w is an injective algebra homomorphism and its image is contained in \mathcal{A}_w . Hence there exists a unique injective linear map

$$(6.5) \quad \Gamma_w : U^{\geq 0}[\hat{T}_w] \rightarrow \mathcal{M}_w$$

such that $\Xi_w \circ \Gamma_w = \Psi_w$.

THEOREM 6.3. *We have*

$$p_i(\mathbf{n}) = d_i(\mathbf{n}) \Gamma_w(\hat{e}_i^{(\mathbf{n})}) \otimes \left\{ \chi_{i_1}^{\langle \beta_{i_1,1}^\vee, \gamma_1 \rangle} \otimes \cdots \otimes \chi_{i_m}^{\langle \beta_{i_m,m}^\vee, \gamma_m \rangle} \right\},$$

where

$$d_i(\mathbf{n}) = \prod_{r=1}^m d_{i_r}(n_r), \quad d_i(n) = q^{n(n+1)/2} (q^{-1} - q)^n.$$

PROOF. For $y \in U^- \cap \hat{T}_w U^-$ we have

$$\begin{aligned} \langle \Xi_w(\Gamma_w(\hat{e}_i^{(\mathbf{n})})) \otimes 1, y \hat{f}_i^{\mathbf{n}'} \rangle &= \varepsilon(y) \langle \Psi_w(\hat{e}_i^{(\mathbf{n})}), \hat{f}_i^{\mathbf{n}'} \rangle = \varepsilon(y) \tau(\hat{e}_i^{(\mathbf{n})}, \hat{f}_i^{\mathbf{n}'}) \\ &= \varepsilon(y) \delta_{\mathbf{n}\mathbf{n}'} \prod_{t=1}^m c_{q_{i_t}}(n_t), \end{aligned}$$

where

$$c_q(n) = [n]! q^{-n(n-1)/2} (q - q^{-1})^{-n}.$$

□

6.4. Set $m_0 = \ell(w_0)$. In this subsection we consider the case $w = w_0$.

LEMMA 6.4. *Let $i \in I$ and define $i' \in I$ by $w_0 \alpha_i = -\alpha_{i'}$. Then we have*

$$\Gamma_{w_0}(e_i) = \frac{1}{1 - q_i^2} (\sigma_{-\varpi_{i'}}^{w_0} e_i) (\sigma_{-\varpi_{i'}}^{w_0})^{-1} \star 1.$$

PROOF. It is sufficient to show

$$\Psi_{w_0}(e_i) = \frac{1}{1 - q_i^2} \Xi_{w_0}((\sigma_{-\varpi_{i'}}^{w_0} e_i) (\sigma_{-\varpi_{i'}}^{w_0})^{-1} \star 1).$$

Set $v^* = v_{-\varpi_{i'}}^* \dot{T}_{w_0}^{-1}$, $v = \dot{T}_{w_0} v_{-\varpi_{i'}}$, so that $v^* \in V^*(-\varpi_{i'})_{\varpi_i}$, $v \in V(-\varpi_{i'})_{\varpi_i}$

with $\langle v^*, v \rangle = 1$. For $t \in U^0$, $y \in U^- \cap \hat{T}_{s_i} U^-$, $p \geq 0$ we have

$$\begin{aligned} \langle \Xi'_{w_0}(\sigma_{-\varpi_{i'}}^{w_0} e_i), ty f_i^p \rangle &= \langle \dot{T}_{w_0}(\sigma_{-\varpi_{i'}}^{w_0} e_i), S(ty f_i^p) \rangle \\ &= \langle v_{-\varpi_{i'}}^* \dot{T}_{w_0}^{-1} e_i, S(ty f_i^p) \dot{T}_{w_0} v_{-\varpi_{i'}} \rangle = \langle v^* e_i, S(ty f_i^p) v \rangle \\ &= \chi_{-\varpi_i}(t) \langle v^* e_i, S(f_i^p) S(y) v \rangle = -\chi_{-\varpi_i}(t) \varepsilon(y) \delta_{p1} \langle v^* e_i, f_i k_i v \rangle \\ &= -\chi_{-\varpi_i}(t) \varepsilon(y) \delta_{p1} q_i \langle v^* e_i, f_i v \rangle \\ &= -\chi_{-\varpi_i}(t) \varepsilon(y) \delta_{p1} q_i \langle v^*, \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}} v \rangle = -\chi_{-\varpi_i}(t) \varepsilon(y) \delta_{p1} q_i. \end{aligned}$$

Hence

$$\begin{aligned} \langle \Xi_{w_0}((\sigma_{-\varpi_{i'}}^{w_0} e_i) (\sigma_{-\varpi_{i'}}^{w_0})^{-1} \star 1), ty f_i^p \rangle &= \langle \chi_{\varpi_i} \Xi'_{w_0}(\sigma_{-\varpi_{i'}}^{w_0} e_i), ty f_i^p \rangle \\ &= \sum_{(t)} \chi_{\varpi_i}(t_{(0)}) (-\chi_{-\varpi_i}(t_{(1)}) \varepsilon(y) \delta_{p1} q_i) = -\varepsilon(t) \varepsilon(y) \delta_{p1} q_i. \end{aligned}$$

On the other hand by (2.8) and Proposition 2.5 we have

$$\langle \Psi_{w_0}(e_i), ty f_i^p \rangle = \tau(e_i, ty f_i^p) = \frac{1}{q_i - q_i^{-1}} \varepsilon(t) \varepsilon(y) \delta_{p1}$$

for $t \in U^0$, $y \in U^- \cap \hat{T}_{s_i} U^-$, $p \geq 0$.

□

PROPOSITION 6.5. For $\mathbf{i} \in \mathcal{I}_{w_0}$, $i \in I$, $\mathbf{n} \in (\mathbb{Z}_{\geq 0})^{m_0}$ write

$$\hat{e}_{\mathbf{i}}^{(\mathbf{n})} e_i = \sum_{\mathbf{n}'} c_{\mathbf{n}\mathbf{n}'} \hat{e}_{\mathbf{i}}^{(\mathbf{n}')}.$$

Then we have

$$\begin{aligned} & \left\{ \frac{1}{1-q_i^2} (\sigma_{-\varpi_{i'}}^{w_0} e_i) (\sigma_{-\varpi_{i'}}^{w_0})^{-1} \right\} p_{\mathbf{i}}(\mathbf{n}) \\ &= \sum_{\mathbf{n}'} c_{\mathbf{n}\mathbf{n}'} \frac{d_{\mathbf{i}}(\mathbf{n})}{d_{\mathbf{i}}(\mathbf{n}')} p_{\mathbf{i}}(\mathbf{n}') \left(\chi_{i_1}^{\langle \beta_{\mathbf{i},1}^\vee, \gamma_{\mathbf{i},\mathbf{n},1} - \gamma_{\mathbf{i},\mathbf{n}',1} \rangle} \otimes \cdots \otimes \chi_{i_{m_0}}^{\langle \beta_{\mathbf{i},m_0}^\vee, \gamma_{\mathbf{i},\mathbf{n},m_0} - \gamma_{\mathbf{i},\mathbf{n}',m_0} \rangle} \right), \end{aligned}$$

where i' is as in Lemma 6.4.

PROOF. Set $\varphi = \frac{1}{1-q_i^2} (\sigma_{-\varpi_{i'}}^{w_0} e_i) (\sigma_{-\varpi_{i'}}^{w_0})^{-1} \in \mathcal{S}_{w_0}^{-1} \mathbb{C}_q[G/N^-]$ so that $\Gamma_{w_0}(e_i) = \varphi \star 1$. For $\mathbf{n}' \in (\mathbb{Z}_{\geq 0})^{m_0}$ take $\varphi_{\mathbf{n}'} \in \mathcal{S}_{w_0}^{-1} \mathbb{C}_q[G/N^-]$ such that $\Gamma_{w_0}(\hat{e}_{\mathbf{i}}^{(\mathbf{n}')}) = \varphi_{\mathbf{n}'} \star 1$. Then we have

$$\begin{aligned} \Xi_{w_0}(\varphi \varphi_{\mathbf{n}} \star 1) &= \Xi_{w_0}(\varphi_{\mathbf{n}} \star 1) \Xi_{w_0}(\varphi \star 1) = \Xi_{w_0}(\Gamma_{w_0}(\hat{e}_{\mathbf{i}}^{(\mathbf{n})})) \Xi_{w_0}(\Gamma_{w_0}(e_i)) \\ &= \Psi_{w_0}(\hat{e}_{\mathbf{i}}^{(\mathbf{n})}) \Psi_{w_0}(e_i) = \Psi_{w_0}(\hat{e}_{\mathbf{i}}^{(\mathbf{n})} e_i) = \sum_{\mathbf{n}'} c_{\mathbf{n}\mathbf{n}'} \Psi_{w_0}(\hat{e}_{\mathbf{i}}^{(\mathbf{n}')}) \\ &= \sum_{\mathbf{n}'} c_{\mathbf{n}\mathbf{n}'} \Xi_{w_0}(\Gamma_{w_0}(\hat{e}_{\mathbf{i}}^{(\mathbf{n}')})) = \sum_{\mathbf{n}'} c_{\mathbf{n}\mathbf{n}'} \Xi_{w_0}(\varphi_{\mathbf{n}'} \star 1). \end{aligned}$$

Hence

$$\varphi \varphi_{\mathbf{n}} \star 1 = \sum_{\mathbf{n}'} c_{\mathbf{n}\mathbf{n}'} \varphi_{\mathbf{n}'} \star 1.$$

It follows that

$$\varphi \Gamma_{w_0}(\hat{e}_{\mathbf{i}}^{(\mathbf{n})}) = \sum_{\mathbf{n}'} c_{\mathbf{n}\mathbf{n}'} \Gamma_{w_0}(\hat{e}_{\mathbf{i}}^{(\mathbf{n}')}).$$

Therefore, the assertion is a consequence of Theorem 6.3. \square

7. SPECIALIZATION

7.1. We denote by $\text{Hom}_{\text{alg}}(\mathbb{C}_q[H], \mathbb{F})$ the set of algebra homomorphisms from $\mathbb{C}_q[H]$ to \mathbb{F} . It is endowed with a structure of commutative group via the multiplication

$$\begin{aligned} (\theta_1 \theta_2)(\chi) &= \sum_{(\chi)} \theta_1(\chi_{(0)}) \theta_2(\chi_{(1)}) \\ &(\theta_1, \theta_2 \in \text{Hom}_{\text{alg}}(\mathbb{C}_q[H], \mathbb{F}), \quad \chi \in \mathbb{C}_q[H]). \end{aligned}$$

The identity element is given by ε , and the inverse of θ is given by $\theta \circ S$.

For $\theta \in \text{Hom}_{\text{alg}}(\mathbb{C}_q[H], \mathbb{F})$ we denote by $\mathbb{F}_\theta = \mathbb{F} 1_\theta$ the corresponding left $\mathbb{C}_q[H]$ -module. For $\theta \in \text{Hom}_{\text{alg}}(\mathbb{C}_q[H], \mathbb{F})$ and $w \in W$ we define an $\mathcal{S}_w^{-1} \mathbb{C}_q[G]$ -module \mathcal{M}_w^θ by

$$\mathcal{M}_w^\theta = \mathcal{M}_w \otimes_{\mathbb{C}_q[H]} \mathbb{F}_\theta.$$

Set $1_w^\theta = (1 \star 1) \otimes 1_\theta \in \mathcal{M}_w^\theta$. We have

$$\mathcal{M}_w^\theta \cong \mathcal{S}_w^{-1} \mathbb{C}_q[G] \otimes_{\mathcal{S}_w^{-1} \mathbb{C}_q[N_w^- \backslash G]} \mathbb{F} \cong \mathbb{C}_q[G] \otimes_{\mathbb{C}_q[N_w^- \backslash G]} \mathbb{F},$$

where $\mathcal{S}_w^{-1} \mathbb{C}_q[N_w^- \backslash G] \rightarrow \mathbb{F}$ is given by $\theta \circ \eta_w$.

Note that we have a decomposition

$$\mathbb{C}_q[N_w^- \setminus G] = \bigoplus_{\lambda \in P} \mathbb{C}_q[N_w^- \setminus G]_\lambda$$

with

$$\mathbb{C}_q[N_w^- \setminus G]_\lambda = \{\varphi \in \mathbb{C}_q[N_w^- \setminus G] \mid t\varphi = \chi_\lambda(t)\varphi \quad (t \in U^0)\}.$$

We have

$$(7.1) \quad (\theta \circ \eta_w)(\varphi) = \varepsilon(\varphi \dot{T}_w) \theta(\chi_\lambda) \quad (\lambda \in P, \varphi \in \mathbb{C}_q[N_w^- \setminus G]_\lambda).$$

Indeed, for $t \in U^0$ we have

$$\langle \eta_w(\varphi), t \rangle = \langle \varphi \dot{T}_w, t \rangle = \langle (t\varphi) \dot{T}_w, 1 \rangle = \chi_\lambda(t) \varepsilon(\varphi \dot{T}_w),$$

and hence $\eta_w(\varphi) = \varepsilon(\varphi \dot{T}_w) \chi_\lambda$. Therefore, $(\theta \circ \eta_w)(\varphi) = \varepsilon(\varphi \dot{T}_w) \theta(\chi_\lambda)$.

The $\mathbb{C}_q[H]$ -module \mathbb{F}_θ can also be regarded as a $\mathbb{C}_q[G]$ -module via the canonical Hopf algebra homomorphism $r_H^G : \mathbb{C}_q[G] \rightarrow \mathbb{C}_q[H]$. We denote this $\mathbb{C}_q[G]$ -module by $\mathbb{F}_\theta^G = \mathbb{F}1_\theta^G$.

PROPOSITION 7.1. *Assume that we are given two algebra homomorphisms $\theta_i : \mathbb{C}_q[H] \rightarrow \mathbb{F}$ ($i = 1, 2$). Then as a $\mathbb{C}_q[G]$ -module we have*

$$\mathcal{M}_w^{\theta_1} \cong \mathcal{M}_w^{\theta_2} \otimes_{\mathbb{F}} \mathbb{F}_{\theta_1(\theta_2 \circ S)}^G.$$

Here, the right side is regarded as a $\mathbb{C}_q[G]$ -module via the comultiplication $\Delta : \mathbb{C}_q[G] \rightarrow \mathbb{C}_q[G] \otimes \mathbb{C}_q[G]$.

PROOF. Let $\lambda \in P$, $\varphi \in \mathbb{C}_q[N_w^- \setminus G]_\lambda$. For $u \in U$, $t \in U^0$ we have

$$\langle \Delta(\varphi), u \otimes t \rangle = \langle \varphi, ut \rangle = \langle t\varphi, u \rangle = \chi_\lambda(t) \langle \varphi, u \rangle,$$

and hence $(\text{id} \otimes r_H^G)(\varphi) = \varphi \otimes \chi_\lambda$. It follows that

$$\begin{aligned} \varphi(1_w^{\theta_2} \otimes 1_{\theta_1(\theta_2 \circ S)}^G) &= (\varphi 1_w^{\theta_2}) \otimes (\chi_\lambda 1_{\theta_1(\theta_2 \circ S)}^G) \\ &= \varepsilon(\varphi \dot{T}_w) \theta_2(\chi_\lambda) (\theta_1(\theta_2 \circ S)) (\chi_\lambda) 1_w^{\theta_2} \otimes 1_{\theta_1(\theta_2 \circ S)}^G \\ &= \varepsilon(\varphi \dot{T}_w) (\theta_2 \theta_1(\theta_2 \circ S)) (\chi_\lambda) 1_w^{\theta_2} \otimes 1_{\theta_1(\theta_2 \circ S)}^G \\ &= \varepsilon(\varphi \dot{T}_w) \theta_1(\chi_\lambda) 1_w^{\theta_2} \otimes 1_{\theta_1(\theta_2 \circ S)}^G = (\theta_1 \circ \eta_w)(\varphi) 1_w^{\theta_2} \otimes 1_{\theta_1(\theta_2 \circ S)}^G. \end{aligned}$$

Hence there exists uniquely a homomorphism $F_{\theta_2}^{\theta_1} : \mathcal{M}_w^{\theta_1} \rightarrow \mathcal{M}_w^{\theta_2} \otimes_{\mathbb{F}} \mathbb{F}_{\theta_1(\theta_2 \circ S)}^G$ of $\mathbb{C}_q[G]$ -modules sending $1_w^{\theta_1}$ to $1_w^{\theta_2} \otimes 1_{\theta_1(\theta_2 \circ S)}^G$. Similarly, we have a homomorphism $F_{\theta_1}^{\theta_2} : \mathcal{M}_w^{\theta_2} \rightarrow \mathcal{M}_w^{\theta_1} \otimes_{\mathbb{F}} \mathbb{F}_{\theta_2(\theta_1 \circ S)}^G$ of $\mathbb{C}_q[G]$ -modules sending $1_w^{\theta_2}$ to $1_w^{\theta_1} \otimes 1_{\theta_2(\theta_1 \circ S)}^G$. Applying $(\bullet) \otimes_{\mathbb{F}} \mathbb{F}_{\theta_1(\theta_2 \circ S)}^G$ to $F_{\theta_1}^{\theta_2}$ we obtain a homomorphism

$$\tilde{F}_{\theta_1}^{\theta_2} := F_{\theta_1}^{\theta_2} \otimes_{\mathbb{F}} \mathbb{F}_{\theta_1(\theta_2 \circ S)}^G : \mathcal{M}_w^{\theta_2} \otimes_{\mathbb{F}} \mathbb{F}_{\theta_1(\theta_2 \circ S)}^G \rightarrow \mathcal{M}_w^{\theta_1}$$

of $\mathbb{C}_q[G]$ -modules sending $1_w^{\theta_2} \otimes 1_{\theta_1(\theta_2 \circ S)}^G$ to $1_w^{\theta_1}$. It remains to show $\tilde{F}_{\theta_1}^{\theta_2} \circ F_{\theta_2}^{\theta_1} = \text{id}$ and $F_{\theta_2}^{\theta_1} \circ \tilde{F}_{\theta_1}^{\theta_2} = \text{id}$. The first identity is a consequence of $(\tilde{F}_{\theta_1}^{\theta_2} \circ F_{\theta_2}^{\theta_1})(1_w^{\theta_1}) = 1_w^{\theta_1}$. The second one follows by applying $(\bullet) \otimes_{\mathbb{F}} \mathbb{F}_{\theta_2(\theta_1 \circ S)}^G$ to $\tilde{F}_{\theta_2}^{\theta_1} \circ F_{\theta_1}^{\theta_2} = \text{id}$. \square

7.2. In view of Proposition 7.1 we only consider the $\mathcal{S}_w^{-1}\mathbb{C}_q[G]$ -module

$$(7.2) \quad \overline{\mathcal{M}}_w = \mathcal{M}_w^\varepsilon$$

in the following. For $\varphi \in \mathcal{S}_w^{-1}\mathbb{C}_q[G]$ we denote by $\overline{\varphi} \in \overline{\mathcal{M}}_w$ the image of $\varphi \star 1$ under $\mathcal{M}_w \rightarrow \overline{\mathcal{M}}_w$. Define $\overline{\eta}_w : \mathcal{S}_w^{-1}\mathbb{C}_q[N_w^- \setminus G] \rightarrow \mathbb{F}$ as the composite $\varepsilon \circ \eta_w$. Then we have

$$\overline{\mathcal{M}}_w \cong \mathcal{S}_w^{-1}\mathbb{C}_q[G] \otimes_{\mathcal{S}_w^{-1}\mathbb{C}_q[N_w^- \setminus G]} \mathbb{F} \cong \mathbb{C}_q[G] \otimes_{\mathbb{C}_q[N_w^- \setminus G]} \mathbb{F},$$

where $\mathcal{S}_w^{-1}\mathbb{C}_q[N_w^- \setminus G] \rightarrow \mathbb{F}$ is given by $\overline{\eta}_w$. Moreover, Θ_w induces a linear isomorphism

$$\overline{\Theta}_w : \overline{\mathcal{M}}_w \rightarrow U^+[\dot{T}_w^{-1}]^\star$$

given by

$$\langle \overline{\Theta}_w(\overline{\varphi}), x \rangle = \langle \varphi \dot{T}_w, x \rangle \quad (\varphi \in \mathbb{C}_q[G], x \in U^+[\dot{T}_w^{-1}]).$$

By the direct sum decomposition

$$U^+[\dot{T}_w^{-1}]^\star = \bigoplus_{\gamma \in Q^+ \cap (-w^{-1}Q^+)} (U^+[\dot{T}_w^{-1}] \cap U_\gamma^+)^*$$

we have a direct sum decomposition

$$(7.3) \quad \overline{\mathcal{M}}_w = \bigoplus_{\gamma \in Q^+ \cap (-w^{-1}Q^+)} \overline{\mathcal{M}}_{w,\gamma},$$

where

$$\overline{\mathcal{M}}_{w,\gamma} = (\overline{\Theta}_w)^{-1}((U^+[\dot{T}_w^{-1}] \cap U_\gamma^+)^*) \quad (\gamma \in Q^+ \cap (-w^{-1}Q^+)).$$

Note that

$$\overline{\mathcal{M}}_{w,0} = \mathbb{F}\mathbf{1}.$$

LEMMA 7.2. *For $m \in \overline{\mathcal{M}}_{w,\gamma}$ and $\lambda \in P$ we have $\sigma_\lambda^w m = q^{-(\lambda,\gamma)} m$.*

PROOF. We may assume $\lambda \in P^-$. Take $\varphi \in \mathbb{C}_q[G]$ such that $\overline{\varphi} = m$. By Corollary 2.7 we have $\dot{T}_w(\sigma_\lambda^w \varphi) = (\dot{T}_w \sigma_\lambda^w)(\dot{T}_w \varphi)$, and hence for $x \in U^+[\dot{T}_w^{-1}]^\star$ we have

$$\langle \overline{\Theta}_w(\sigma_\lambda^w m), x \rangle = \langle \dot{T}_w(\sigma_\lambda^w \varphi), \dot{T}_w(x) \rangle = \langle (\dot{T}_w \sigma_\lambda^w)(\dot{T}_w \varphi), \dot{T}_w(x) \rangle.$$

Assume $x \in U^+[\dot{T}_w^{-1}] \cap U_\delta^+$ with $\delta \in Q^+ \cap (-w^{-1}Q^+)$, and set $y = \dot{T}_w(x)$. Then we have $y \in (U_{w\delta}^-)k_{-w\delta}$ by (2.35). Hence

$$\begin{aligned} \langle \overline{\Theta}_w(\sigma_\lambda^w m), x \rangle &= \sum_{(y)} \langle \dot{T}_w \sigma_\lambda^w, y_{(0)} \rangle \langle \dot{T}_w \varphi, y_{(1)} \rangle = \langle \dot{T}_w \sigma_\lambda^w, k_{-w\delta} \rangle \langle \dot{T}_w \varphi, y \rangle \\ &= q^{-(\lambda,\delta)} \langle \overline{\Theta}_w(m), x \rangle. \end{aligned}$$

□

THEOREM 7.3. *The $\mathbb{C}_q[G]$ -module $\overline{\mathcal{M}}_w$ is irreducible.*

PROOF. For any $\mathbb{C}_q[G]$ -submodule N of $\overline{\mathcal{M}}_w$ we have

$$N = \bigoplus_{\gamma \in Q^+ \cap (-w^{-1}Q^+)} (N \cap \overline{\mathcal{M}}_{w,\gamma})$$

by Lemma 7.2. Since $\overline{\mathcal{M}}_{w,0}$ is one-dimensional and generates the $\mathbb{C}_q[G]$ -module $\overline{\mathcal{M}}_w$, it is sufficient to show $N \cap \overline{\mathcal{M}}_{w,0} \neq \{0\}$ for any non-zero

$\mathbb{C}_q[G]$ -submodule N of $\overline{\mathcal{M}}_w$. By definition the projection $\overline{\mathcal{M}}_w \rightarrow \overline{\mathcal{M}}_{w,0}$ with respect to (7.3) is given by $m \mapsto \langle \overline{\Theta}_w(m), 1 \rangle \overline{1}$, and hence it is sufficient to show that for any $m \in \overline{\mathcal{M}}_w \setminus \{0\}$ there exists some $\psi \in \mathbb{C}_q[G]$ such that $\langle \overline{\Theta}_w(\psi m), 1 \rangle \neq 0$. Take $\varphi \in \mathbb{C}_q[G]$ such that $\overline{\varphi} = m$. For $z \in U^+$ and $\lambda \in P^-$ we have

$$\langle \overline{\Theta}_w((z\sigma_\lambda^w)m), 1 \rangle = \langle \{(z\sigma_\lambda^w)\varphi\} \dot{T}_w, 1 \rangle.$$

Write

$$\Delta \dot{T}_w = (\dot{T}_w \otimes \dot{T}_w) \sum_j u_j^- \otimes u_j^+$$

(see Corollary 2.7). Then we have

$$\begin{aligned} \langle \overline{\Theta}_w((z\sigma_\lambda^w)m), 1 \rangle &= \sum_j \langle z\sigma_\lambda^w \dot{T}_w u_j^-, 1 \rangle \langle \varphi \dot{T}_w u_j^+, 1 \rangle \\ &= \sum_j \langle v_\lambda^* u_j^-, z v_\lambda \rangle \langle \overline{\Theta}_w(m), u_j^+ \rangle = \langle v_\lambda^* y z, v_\lambda \rangle \end{aligned}$$

with $y = \sum_j \langle \overline{\Theta}_w(m), u_j^+ \rangle u_j^- \in U^- \setminus \{0\}$. Hence it is sufficient to show that for any $y \in U^-$ there exists some $\lambda \in P^-$ and $z \in U^+$ such that $\langle v_\lambda^* y z, v_\lambda \rangle \neq 0$. If $\lambda \in P^-$ is sufficiently small, then we have $v_\lambda^* y \neq 0$. Then the assertion is a consequence of the irreducibility of $V^*(\lambda)$ as a right U -module. \square

7.3. For $i \in I$ we define a $\mathbb{C}_q[G]$ -module $\overline{\mathcal{M}}_i$ by

$$(7.4) \quad \overline{\mathcal{M}}_i = \mathcal{M}_i \otimes_{\mathbb{C}_q[H(i)]} \mathbb{F}_\varepsilon.$$

It is an irreducible $\mathbb{C}_q[G]$ -module with basis $\{\overline{p}_i(n) \mid n \in \mathbb{Z}_{\geq 0}\}$ satisfying

$$\begin{aligned} a_i \overline{p}_i(n) &= (1 - q_i^{2n}) \overline{p}_i(n-1), & b_i \overline{p}_i(n) &= q_i^n \overline{p}_i(n), \\ c_i \overline{p}_i(n) &= -q_i^{n+1} \overline{p}_i(n) i, & d_i \overline{p}_i(n) &= \overline{p}_i(n+1). \end{aligned}$$

Fix $w \in W$, and set $\ell(w) = m$. For $\mathbf{i} = (i_1, \dots, i_m) \in \mathcal{I}_w$ the isomorphism (5.6) induces an isomorphism

$$(7.5) \quad \overline{\mathcal{M}}_w \cong \overline{\mathcal{M}}_{i_1} \otimes \dots \otimes \overline{\mathcal{M}}_{i_m}$$

of $\mathbb{C}_q[G]$ -modules. For $\mathbf{n} = (n_1, \dots, n_m) \in (\mathbb{Z}_{\geq 0})^m$ we denote by $\overline{p}_i(\mathbf{n})$ the element of $\overline{\mathcal{M}}_w$ corresponding to

$$\overline{p}_{i_1}(n_1) \otimes \dots \otimes \overline{p}_{i_m}(n_m) \in \overline{\mathcal{M}}_{i_1} \otimes \dots \otimes \overline{\mathcal{M}}_{i_m}$$

via the isomorphism (7.5).

By Theorem 6.3 we have the following.

THEOREM 7.4. *For $\mathbf{i}, \mathbf{j} \in \mathcal{I}_w$ we have*

$$\hat{e}_j^{(\mathbf{n})} = \sum_{\mathbf{n}'} a_{\mathbf{n}'} \hat{e}_i^{(\mathbf{n}')} \implies \overline{p}_j(\mathbf{n}) = \sum_{\mathbf{n}'} a_{\mathbf{n}'} \frac{d_{j,\mathbf{n}}}{d_{i,\mathbf{n}'}} \overline{p}_i(\mathbf{n}'),$$

where $d_{i,\mathbf{n}}$ is as in Theorem 6.3.

REMARK 7.5. Theorem 7.4 for $w = w_0$ is the main result of Kuniba, Okado, Yamada ([8, Theorem 5])

By Proposition 6.5 we have the following.

PROPOSITION 7.6. For $\mathbf{i} \in \mathcal{I}_{w_0}$, $i \in I$, $\mathbf{n} \in (\mathbb{Z}_{\geq 0})^{m_0}$ write

$$\hat{e}_{\mathbf{i}}^{(\mathbf{n})} e_i = \sum_{\mathbf{n}'} c_{\mathbf{n}\mathbf{n}'} \hat{e}_{\mathbf{i}}^{(\mathbf{n}')}.$$

Then we have

$$\left\{ \frac{1}{1 - q_i^2} (\sigma_{-\varpi_{i'}}^{w_0} e_i) (\sigma_{-\varpi_{i'}}^{w_0})^{-1} \right\} \bar{p}_{\mathbf{i}}(\mathbf{n}) = \sum_{\mathbf{n}'} c_{\mathbf{n}\mathbf{n}'} \frac{d_{\mathbf{i}}(\mathbf{n})}{d_{\mathbf{i}}(\mathbf{n}')} \bar{p}_{\mathbf{i}}(\mathbf{n}'),$$

where i' is as in Lemma 6.4.

REMARK 7.7. Proposition 7.6 is a conjecture of Kuniba, Okado, Yamada ([8, Conjecture 1]).

8. COMMENTS

8.1. In this paper we worked over the base field $\mathbb{F} = \mathbb{Q}(q)$; however, almost all of the arguments work equally well after minor modifications even when \mathbb{F} is an arbitrary field of characteristic zero and $\overline{q_i^2} \neq 1$ for any $i \in I$. The only exception is Theorem 7.3, which states that $\overline{\mathcal{M}}_w$ is irreducible. For this result we need to assume that q is not a root of 1.

8.2. Let us consider generalization of our results to the case where \mathfrak{g} is a symmetrizable Kac-Moody Lie algebra. We take $\mathbb{C}_q[G]$ to be the subspace of $U_q(\mathfrak{g})^*$ spanned by the matrix coefficients of integrable lowest weight modules (see [5]). Then $\mathbb{C}_q[G]$ is naturally endowed with an algebra structure. A problem is that the comultiplication $\Delta : \mathbb{C}_q[G] \rightarrow \mathbb{C}_q[G] \otimes \mathbb{C}_q[G]$ is not defined. Indeed $\Delta(\varphi)$ for $\varphi \in \mathbb{C}_q[G]$ turns out to be an infinite sum which belongs to a completion of $\mathbb{C}_q[G] \otimes \mathbb{C}_q[G]$. However, since we only consider the tensor product modules of type $\mathcal{M}_{i_1} \otimes \cdots \otimes \mathcal{M}_{i_m}$, what we actually need is the homomorphism of the form

$$(8.1) \quad (r_{G(i_1)}^G \otimes \cdots \otimes r_{G(i_m)}^G) \circ \Delta_{m-1} : \mathbb{C}_q[G] \rightarrow \mathbb{C}_q[G(i_1)] \otimes \cdots \otimes \mathbb{C}_q[G(i_m)].$$

We can easily check that (8.1) is well-defined even in the Kac-Moody setting by showing that $(r_{G(i_1)}^G \otimes \cdots \otimes r_{G(i_m)}^G) \circ \Delta_{m-1}$ sends any element of $\mathbb{C}_q[G]$ to a finite sum inside $\mathbb{C}_q[G(i_1)] \otimes \cdots \otimes \mathbb{C}_q[G(i_m)]$. It is easily seen that all of the arguments in this paper also work in the setting where \mathfrak{g} is a symmetrizable Kac-Moody Lie algebra.

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